Asymptotic behavior of solutions of the stationary Navier-Stokes equations in an exterior domain

Ching-Lung Lin* Gunther Uhlmann† Jenn-Nan Wang‡

Abstract

We study the asymptotic behavior of an incompressible fluid around a bounded obstacle. The problem is modelled by the stationary Navier-Stokes equations in an exterior domain in $\mathbb{R}^n$ with $n \geq 2$. We will show that, under some assumptions, any nontrivial velocity field obeys a minimal decaying rate $\exp(-Ct^2 \log t)$ at infinity. Our proof is based on appropriate Carleman estimates.

1 Introduction

Let B be a bounded domain in $\mathbb{R}^n$ and $\Omega = \mathbb{R}^n \setminus \bar{B}$ with $n \geq 2$. Without loss of generality, we assume 0 is in the interior of B and $B \subset B_1(0) = \{x : |x| < 1\}$. Assume that Ω is filled with an incompressible fluid described by the stationary Navier-Stokes equations

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in} \quad \Omega, \\ \nabla \cdot u = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

*Department of Mathematics, NCTS, National Cheng Kung University, Tainan 701, Taiwan. Email:cllin2@mail.ncku.edu.tw

†Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA and Department of Mathematics University of California, Irvine, CA 92697-3875, USA. Email:gunther@math.washington.edu

‡Department of Mathematics, Taida Institute of Mathematical Sciences, NCTS (Taipei), National Taiwan University, Taipei 106, Taiwan. Email: jn-wang@math.ntu.edu.tw

We are interested in the following question: under some boundedness assumption on u, what is the minimal decaying rate at infinity of any nontrivial u satisfying (1.1)?

To put the problem in perspective, we first mention some related results. In three dimensions, Finn [3] showed that if $u|_{\partial B}=0$ and $u=o(|x|^{-1})$, then u is trivial. In the same setting and assuming, additionally, that u is C^2 bounded, Dyer and Edmunds [2] proved that if $u=O(\exp(-\exp(\alpha|x|^3)))$ for all $\alpha>0$, then u is trivial. We remark that in Finn's result u is required to satisfy the homogeneous Dirichlet condition on ∂B and a decaying condition at infinity, while in Dyer and Edmunds's result, with the assumption of C^2 boundedness, only the local behavior of u at infinity is needed. We showed in an early paper [6] that for $n=2$ or 3, if u is bounded in Ω, then any nontrivial u of (1.1) can not decay faster than certain double exponential at infinity (see [6, Corollary 1.6] for details). In the present paper, we improve significantly on that result, and the result of [2] by showing that the minimal decaying rate of any nontrivial u is close to exponential in dimension $n\ge 2$. We now state the main theorem of the paper.

Denote

$$M(t)=\inf_{|x|=t}\int_{|y-x|<1}|u(y)|^2dy.$$

Theorem 1.1 *Let $u\in(H^1_{loc}(\Omega))^n$ be a nontrivial solution of (1.1) with an appropriate $p\in H^1_{loc}(\Omega)$. Assume that*

$$\|u\|_{L^\infty(\Omega)}\le\lambda\quad\text{if}\quad n=2,\tag{1.2}$$

or

$$\|u\|_{L^\infty(\Omega)}+\|\nabla u\|_{L^\infty(\Omega)}\le\lambda\quad\text{if}\quad n\ge 3.\tag{1.3}$$

Then there exist $C>0$ depending on λ, n, and $\tilde t>0$ depending on λ, n, $M(10)$ such that

$$M(t)\ge\exp(-Ct^2\log t)\quad\text{for}\quad t\ge\tilde t.$$

Remark 1.2 *It is interesting to compare our result with a similar result for the Schrödinger equation proved by Bourgain and Kenig [1] (see also [5]). In [1], they considered the Schrödinger equation*

$$\Delta u+V(x)u=0\quad\text{in}\quad\mathbb{R}^n.$$

Under the assumption that $|V| \leq 1$, $|u_0| \leq C_0$, and $u(0) = 1$, they proved

$$\inf_{|x_0|=R} \sup_{B(x_0,1)} |u(x)| \geq C \exp(-R^{4/3} \log R) \quad \textit{for} \quad R >> 1.$$

We prove our results by using appropriate Carleman estimates. We will use weights which are slightly less singular than negative powers of $|x|$ (see estimates (2.1)). The method of obtaining a decaying rate is a detour from that of deriving three-ball inequalities using Carleman estimates.

This paper is organized as follows. In Section 2, we reduce the Navier-Stokes equations to a new system by using the vorticity function. Then we state some suitable Carleman-type estimates. A technical interior estimate is proved in Section 3. Section 4 is devoted to the proof of Theorem 1.1.

2 Reduced system and Carleman estimates

Fixing x_0 with $|x_0| = t >> 1$, we define

$$w(x) = (at)u(atx + x_0), \quad \tilde{p}(x) = (at)^2 p(atx + x_0),$$

where $a = 8/s$ and $0 < s < 8$ is a small constant which will be determined in the proof of Theorem 1.1. Likewise, we denote

$$\Omega_t := B_{\frac{1}{a} - \frac{1}{at}}(0).$$

From (1.1), it is easy to check that

$$\begin{cases} -\Delta w + w \cdot \nabla w + \nabla \tilde{p} = 0 \quad \text{in} \quad \Omega_t, \\ \nabla \cdot w = 0 \quad \text{in} \quad \Omega_t. \end{cases} \tag{2.1}$$

In view of (1.2) and (1.3), we have that

$$\|w\|_{L^\infty(\Omega_t)} \leq at\lambda \tag{2.2}$$

or

$$\|w\|_{L^\infty(\Omega_t)} \leq at\lambda \quad \text{and} \quad \|\nabla w\|_{L^\infty(\Omega_t)} \leq (at)^2\lambda. \tag{2.3}$$

To study the Navier-Stokes equation, it is often advantageous to consider the vorticity equation. Let us now define the vorticity q of the velocity w by

$$q = \text{curl} w := \frac{1}{\sqrt{2}}(\partial_i w_j - \partial_j w_i)_{1 \leq i,j \leq n}.$$

Note that here q is a matrix-valued function. The formal transpose of curl is given by

$$(\mathrm{curl}^\top v)_{1\le i\le n} := \frac{1}{\sqrt{2}} \sum_{1\le j\le n} \partial_j(v_{ij} - v_{ji}),$$

where $v = (v_{ij})_{1\le i,j\le n}$. It is easy to see that

$$\Delta w = \nabla(\nabla \cdot w) - \mathrm{curl}^\top \mathrm{curl} w$$

(see, for example, [8] for a proof), which implies

$$\Delta w + \mathrm{curl}^\top q = 0 \quad \text{in} \quad \Omega_t.$$

On the other hand, we observe that

$$w \cdot \nabla w = \nabla(\frac{1}{2}|w|^2) - \sqrt{2}(\mathrm{curl} w)w = \nabla(\frac{1}{2}|w|^2) - \sqrt{2} q w.$$

Thus, applying curl on the first equation of (2.2), we have that

$$-\Delta q + Q(q)w + q(\nabla w)^\top - (\nabla w)q^\top = 0 \quad \text{in} \quad \Omega_t,$$

where

$$(Q(q)w)_{ij} = \sum_{1\le k\le n} (\partial_j q_{ik} - \partial_i q_{jk}) w_k.$$

Now for $n = 2$, due to $\nabla \cdot w = 0$, it is easily seen that

$$q(\nabla w)^\top - (\nabla w)q^\top = 0.$$

Therefore, we will consider the system

$$\begin{cases} -\Delta q + Q(q)w + q(\nabla w)^\top - (\nabla w)q^\top = 0 \quad \text{in} \quad \Omega_t, \\ \Delta w + \mathrm{curl}^\top q = 0 \quad \text{in} \quad \Omega_t \end{cases} \tag{2.4}$$

for $n \ge 3$, and

$$\begin{cases} -\Delta q + Q(q)w = 0 \quad \text{in} \quad \Omega_t, \\ \Delta w + \mathrm{curl}^\top q = 0 \quad \text{in} \quad \Omega_t \end{cases} \tag{2.5}$$

for $n = 2$. In order to prove the main theorem, putting together (2.4), (2.5), and using (2.2), (2.3), it suffices to consider

$$\begin{cases} -\Delta q + A(x) \cdot \nabla q + B(x) q = 0 \quad \text{in} \quad \Omega_t, \\ \Delta w + \mathrm{curl}^\top q = 0 \quad \text{in} \quad \Omega_t \end{cases} \tag{2.6}$$

with

$$\|A\|_{L^\infty(\Omega_t)} \le at\lambda \quad \text{and} \quad \|B\|_{L^\infty(\Omega_t)} \le (at)^2\lambda.$$

For our proof, we will apply Carleman estimates with weights $\varphi_\beta = \varphi_\beta(x) = \exp(-\beta\tilde{\psi}(x))$, where $\beta > 0$ and $\tilde{\psi}(x) = \log|x| + \log((\log|x|)^2)$.

Lemma 2.1 *There exist a sufficiently small number $r_1 > 0$, a sufficiently large number $\beta_1 > 2$, a positive constant C, such that for all $v \in U_{r_1}$ and $\beta \ge \beta_1$, we have that*

$$\int \varphi_\beta^2(\log|x|)^2(\beta|x|^{4-n}|\nabla v|^2 + \beta^3|x|^{2-n}|v|^2)dx \le C\int \varphi_\beta^2(\log|x|)^4|x|^{6-n}|\Delta v|^2dx, \tag{2.7}$$

where $U_{r_1} = \{v \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \mathrm{supp}(v) \subset B_{r_1}\}$.

Lemma 2.1 is a modified form of [7, Lemma 2.4]. For the sake of brevity, we omit the proof here. Applying Lemma 2.1 with $\beta = \beta + 1$, we have the following Carleman estimates.

Lemma 2.2 *There exist a sufficiently small number $r_1 > 0$, a sufficiently large number $\beta_1 > 1$, a positive constant C, such that for all $v \in U_{r_1}$ and $\beta \ge \beta_1$, we have*

$$\int \varphi_\beta^2(\log|x|)^{-2}|x|^{-n}(\beta|x|^2|\nabla v|^2 + \beta^3|v|^2)dx \le C\int \varphi_\beta^2|x|^{-n}(|x|^4|\Delta v|^2)dx. \tag{2.8}$$

3 Interior estimates

In addition to Carleman estimates, we also need the following interior inequality.

Lemma 3.1 *For any $0 < a_1 < a_2$ such that $B_{a_2} \subset \Omega_t$, let $X = B_{a_2}\backslash\bar{B}_{a_1}$ and $d(x)$ be the distance from $x \in X$ to $\mathbb{R}^n\backslash X$. We have*

$$\begin{aligned}
&\int_X d(x)^2|\nabla w|^2dx + \int_X d(x)^4|\nabla q|^2dx + \int_X d(x)^2|q|^2dx \\
\le\ & C(at)^{12}\int_X |w|^2dx.
\end{aligned} \tag{3.1}$$

where the constant C is independent of r, a, t and (w, q).

Proof. By elliptic regularity, we obtain from (1.1) that $u \in H^2_{loc}(\Omega)$ and hence $w \in H^2(\Omega_t)$. It is trivial that

$$\|v\|_{H^1(\mathbb{R}^n)} \lesssim \|\Delta v\|_{L^2(\mathbb{R}^n)} + \|v\|_{L^2(\mathbb{R}^n)} \tag{3.2}$$

for all $v \in H^2(\mathbb{R}^n)$. By changing variables $x \to E^{-1}x$ in (3.2), we obtain

$$\textstyle\sum_{|\alpha|\le 1} E^{2-|\alpha|}\|D^\alpha v\|_{L^2(\mathbb{R}^n)} \lesssim (\|\Delta v\|_{L^2(\mathbb{R}^n)} + E^2\|v\|_{L^2(\mathbb{R}^n)}) \tag{3.3}$$

for all $v \in H^2(\mathbb{R}^n)$. To apply (3.3) to w, we need to cut-off w. So let $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \xi(x) \le 1$ and

$$\xi(x) = \begin{cases} 1, & |x| < 1/4, \\ 0, & |x| \ge 1/2. \end{cases}$$

Let us denote $\xi_y(x) = \xi((x-y)/d(y))$. For $y \in X$, we apply (3.3) to $\xi_y(x)w(x)$ and use the second equation of (2.6) to get that

$$\begin{aligned} & E^2 \int_{|x-y|\le d(y)/4} |\nabla w|^2 dx \\ \le\ & C_1 \int_{|x-y|\le d(y)/2} |\nabla q|^2 dx + C_1 \int_{|x-y|\le d(y)/2} d(y)^{-2}|\nabla w|^2 dx \\ & + C_1(E^4 + d(y)^{-4}) \int_{|x-y|\le d(y)/2} |w|^2 dx. \end{aligned} \tag{3.4}$$

Now taking $E = M^3 d(y)^{-1}$ for some constant $M > 1$ and multiplying $d(y)^4$ on both sides of (3.4), we have

$$\begin{aligned} & M^6 d(y)^2 \int_{|x-y|\le d(y)/4} |\nabla w|^2 dx \\ \le\ & C_1 \int_{|x-y|\le d(y)/2} d(y)^4 |\nabla q|^2 dx + C_1 \int_{|x-y|\le d(y)/2} d(y)^2|\nabla w|^2 dx \\ & + C_1(M^{12} + 1) \int_{|x-y|\le d(y)/2} |w|^2 dx. \end{aligned} \tag{3.5}$$

Integrating $d(y)^{-n}dy$ over X on both sides of (3.5) and using Fubini's Theorem, we get that

$$\begin{aligned} & M^6 \int_X \int_{|x-y|\le d(y)/4} d(y)^{2-n}|\nabla w|^2 dydx \\ \le\ & C_1 \int_X \int_{|x-y|\le d(y)/2} d(y)^{4-n}|\nabla q(x)|^2 dydx \\ & +C_1 \int_X \int_{|x-y|\le d(y)/2} d(y)^{2-n}|\nabla w|^2 dydx \\ & +2C_1 M^{12} \int_X \int_{|x-y|\le d(y)/2} d(y)^{-n}|w|^2 dydx. \end{aligned} \tag{3.6}$$

Note that $|d(x)-d(y)| \le |x-y|$. If $|x-y| \le d(x)/3$, then

$$2d(x)/3 \le d(y) \le 4d(x)/3. \tag{3.7}$$

On the other hand, if $|x-y| \le d(y)/2$, then

$$d(x)/2 \le d(y) \le 3d(x)/2. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\begin{cases} \int_{|x-y|\le d(y)/4} d(y)^{-n}dy \ge (3/4)^n \int_{|x-y|\le d(x)/6} d(x)^{-n}dy \ge 8^{-n} \int_{|y|\le 1} dy, \\ \int_{|x-y|\le d(y)/2} d(y)^{-n}dy \le 2^n \int_{|x-y|\le 3d(x)/4} d(x)^{-n}dy \le (3/2)^n \int_{|y|\le 1} dy. \end{cases} \tag{3.9}$$

Combining (3.6)–(3.9), we obtain

$$\begin{aligned} & M^6 \int_X d(x)^2|\nabla w|^2 dx \\ \le\ & C_2 \int_X d(x)^2|\nabla w(x)|^2 dx + C_2 \int_X d(x)^4|\nabla q|^2 dx + C_2 M^{12} \int_X |w|^2 dx. \end{aligned} \tag{3.10}$$

On the other hand, we have from the first equation of (2.6) that

$$\begin{aligned} & E^2 \int_{|x-y|\le d(y)/4} |\nabla q|^2 dx \\ \le\ & C_3((at)^2 + d(y)^{-2}) \int_{|x-y|\le d(y)/2} |\nabla q|^2 dx \\ & +C_3(E^4 + d(y)^{-4} + (at)^4) \int_{|x-y|\le d(y)/2} |q|^2 dx. \end{aligned} \tag{3.11}$$

Now taking $E = Md(y)^{-1}$ and multiplying $d(y)^6$ on both sides of (3.4), we have

$$
\begin{aligned}
& M^2 d(y)^4 \int_{|x-y|\le d(y)/4} |\nabla q|^2 dx \\
\le\ & C_3((at)^2 d(y)^2 + 1) \int_{|x-y|\le d(y)/2} d(y)^4 |\nabla q|^2 dx \\
& + C_3(M^4 + 1 + (at)^4 d(y)^4) \int_{|x-y|\le d(y)/2} d(y)^2 |q|^2 dx. \qquad (3.12)
\end{aligned}
$$

Repeating the arguments in (3.6)∼(3.10), we have that

$$
\begin{aligned}
& M^2 \int_X d(x)^4 |\nabla q|^2 dx \\
\le\ & C_4 \int_X ((at)^2 d(x)^2 + 1) d(x)^4 |\nabla q|^2 dx \\
& + C_4 \int_X (M^4 + 1 + (at)^4 d(x)^4) d(x)^2 |q|^2 dx \\
\le\ & C_5 \int_X ((at)^2 d(x)^2 + 1) d(x)^4 |\nabla q|^2 dx \\
& + C_5 \int_X (M^4 + 1 + (at)^4 d(x)^4) d(x)^2 |\nabla w|^2 dx. \qquad (3.13)
\end{aligned}
$$

Combining (3.13) and (3.10), we obtain that if $M \ge M_0$ for some $M_0 > 1$ then

$$
\begin{aligned}
& M^4 \int_X d(x)^2 |\nabla w|^2 dx + M^2 \int_X d(x)^4 |\nabla q|^2 dx \\
\le\ & C_6 \int_X ((at)^4 d(x)^4) d(x)^2 |\nabla w(x)|^2 dx \\
& + C_6 M^{12} \int_X |w|^2 dx + C_6 \int_X ((at)^2 d(x)^2) d(x)^4 |\nabla q|^2 dx. \qquad (3.14)
\end{aligned}
$$

Note that $B_{a_2} \subset \Omega_t$ and therefore

$$d(x) < \frac{1}{a} - \frac{1}{at} < 1.$$

Taking $M = (C_6+1)at$, one can eliminate $\int_X d(x)^4|\nabla q|^2 dx$ and $\int_X d(x)^2|\nabla w(x)|^2 dx$ on the right hand side of (3.14). Finally, we get that

$$\begin{aligned} & (at)^4 \int_X d(x)^2|\nabla w|^2 dx + \int_X d(x)^4|\nabla q|^2 dx \\ \leq \ & C_7(at)^{12} \int_X |w|^2 dx. \end{aligned} \tag{3.15}$$

It is no harm to add $\int_X d(x)^2|q|^2 dx$ to the right hand side of (3.15) since $\int_X d(x)^2|q|^2 dx \leq \int_X d(x)^2|\nabla w|^2 dx$. We then obtain (3.1). $\square$

4 Proof of Theorem 1.1

This section is devoted to the proof of the main theorem, Theorem 1.1. Since $(w,p) \in (H^1(\Omega_t))^{n+1}$, the regularity theorem implies $w \in H^2_{loc}(\Omega_t)$. Therefore, to use estimate (2.7), we simply cut-off w. So let $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \chi(x) \leq 1$ and

$$\chi(x) = \begin{cases} 0, & |x| \leq \frac{1}{4at}, \\ 1, & \frac{1}{2at} < |x| < \frac{1}{a} - \frac{3}{at}, \\ 0, & |x| \geq \frac{1}{a} - \frac{2}{at}. \end{cases}$$

It is easy to see that for any multiindex α

$$\begin{cases} |D^\alpha \chi| = O((at)^{|\alpha|}) & \text{if } \ \frac{1}{4at} \leq |x| \leq \frac{1}{2at}, \\ |D^\alpha \chi| = O((at)^{|\alpha|}) & \text{if } \ \frac{1}{a} - \frac{3}{at} \leq |x| \leq \frac{1}{a} - \frac{2}{at}. \end{cases} \tag{4.1}$$

If we choose $s < 8r_1$, then $\operatorname{supp}(\chi) \subset B_{r_1}$, where r_1 is defined in Lemma 2.1. Therefore, applying (2.8) to χw gives

$$\begin{aligned} & \int (\log|x|)^{-2} \varphi_\beta^2 |x|^{-n} (\beta |x|^2 |\nabla(\chi w)|^2 + \beta^3 |\chi w|^2) dx \\ \leq \ & C \int \varphi_\beta^2 |x|^{-n} |x|^4 |\Delta(\chi w)|^2 dx. \end{aligned} \tag{4.2}$$

Here and after, C and $\tilde{C}$ denote general constants whose value may vary from line to line. The dependence of C and $\tilde{C}$ will be specified whenever necessary. Next applying (2.7) to $v = \chi q$ yields that

$$\begin{aligned}
& \int \varphi_\beta^2 (\log|x|)^2 (|x|^{4-n}\beta|\nabla(\chi q)|^2 + |x|^{2-n}\beta^3|\chi q|^2)dx \\
\leq \;& C\int \varphi_\beta^2 (\log|x|)^4|x|^{6-n}|\Delta(\chi q)|^2 dx. \qquad (4.3)
\end{aligned}$$

Combining (4.2) and (4.3), we obtain that

$$\begin{aligned}
& \int_W (\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(\beta|x|^2|\nabla w|^2 + \beta^3|w|^2)dx \\
& + \int_W (\log|x|)^2\varphi_\beta^2|x|^{-n}(\beta|x|^4|\nabla q|^2 + |x|^2\beta^3|q|^2)dx \\
\leq \;& \int \varphi_\beta^2(\log|x|)^{-2}|x|^{-n}(\beta|x|^2\nabla(\chi w)|^2 + \beta^3|\chi w|^2)dx \\
& + \int (\log|x|)^2\varphi_\beta^2|x|^{-n}(\beta|x|^4|\nabla(\chi q)|^2 + \beta^3|x|^2|\chi q|^2)dx \\
\leq \;& C\int \varphi_\beta^2|x|^{-n}|x|^4|\Delta(\chi w)|^2 dx \\
& + C\int \varphi_\beta^2(\log|x|)^4|x|^{6-n}|\Delta(\chi q)|^2 dx, \qquad (4.4)
\end{aligned}$$

where $W = \{x : \frac{1}{2at} < |x| < \frac{1}{a} - \frac{3}{at}\}$. Define $Y = \{x : \frac{s}{32t} = \frac{1}{4at} \leq |x| \leq \frac{1}{2at} = \frac{s}{16t}\}$ and $Z = \{x : \frac{1}{a} - \frac{3}{at} \leq |x| \leq \frac{1}{a} - \frac{2}{at}\}$. By (2.6) and estimates (4.1), we deduce from (4.4) that

$$\begin{aligned}
& \int_W (\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(\beta|x|^2|\nabla w|^2 + \beta^3|w|^2)dx \\
& + \int_W (\log|x|)^2\varphi_\beta^2|x|^{-n}(\beta|x|^4|\nabla q|^2 + |x|^2\beta^3|q|^2)dx \\
\leq \;& C\int_W \varphi_\beta^2|x|^{-n}|x|^4|\nabla q|^2 dx \\
& + C\int_W (\log|x|)^4\varphi_\beta^2|x|^{-n}|x|^6((at)^4|q|^2 + (at)^2|\nabla q|^2)dx \\
& + C(at)^4\int_{Y\cup Z} \varphi_\beta^2|x|^{-n}|\tilde{U}|^2 dx \\
& + C(at)^4\int_{Y\cup Z} (\log|x|)^4\varphi_\beta^2|x|^{2-n}|\tilde{U}|^2 dx, \qquad (4.5)
\end{aligned}$$

where $|\tilde{U}(x)|^2 = |x|^4|\nabla q|^2+|x|^2|q|^2+|x|^2|\nabla w|^2+|w|^2$ and the positive constant C only depends on λ and n.

It is easy to check that there exists $\tilde{R}_1 > 0$, depending on n, such that for all $\beta > 0$, both $(\log|x|)^{-2}|x|^{-n}\varphi_\beta^2(|x|)$ and $(\log|x|)^4|x|^{-n}\varphi_\beta^2(|x|)$ are decreasing functions in $0 < |x| < \tilde{R}_1$. So we choose a small $s < 8\min\{r_1, \tilde{R}_1\}$. Now letting $\beta \geq \tilde{\beta}$ with $\tilde{\beta} = C(at)^2 + 1$, then the first two terms on the right hand side of (4.5) can be absorbed by the left hand side of (4.5). With the choices described above, we obtain from (4.5) that

$$\begin{aligned}
& \beta^3(b_1)^{-n}(\log b_1)^{-2}\varphi_\beta^2(b_1)\int_{\frac{1}{at}<|x|<b_1}|w|^2dx \\
\leq\ & \beta^3\int_W(\log|x|)^{-2}\varphi_\beta^2|x|^{-n}|w|^2dx \\
\leq\ & C(at)^4\int_{Y\cup Z}(\log|x|)^4\varphi_\beta^2|x|^{-n}|\tilde{U}|^2dx \\
\leq\ & C(at)^4(\log b_2)^4 b_2^{-n}\varphi_\beta^2(b_2)\int_Y|\tilde{U}|^2dx \\
& +C(at)^4(\log b_3)^4 b_3^{-n}\varphi_\beta^2(b_3)\int_Z|\tilde{U}|^2dx,
\end{aligned} \tag{4.6}$$

where $b_1 = \frac{1}{a} - \frac{8}{at}$, $b_2 = \frac{1}{4at}$ and $b_3 = \frac{1}{a} - \frac{3}{at}$.

Using (3.1), we can control the $|\tilde{U}|^2$ terms on the right hand side of (4.5). Indeed, let $X = Y_1 := \{x : \frac{1}{8at} \leq |x| \leq \frac{1}{at}\}$, then we can see that

$$d(x) \geq C|x| \quad \text{for all} \quad x \in Y,$$

where C an absolute constant. Therefore, (3.1) implies

$$\begin{aligned}
& \int_Y\left(|x|^2|\nabla w|^2+|x|^4|\nabla q|^2+|x|^2|q|^2\right)dx \\
\leq\ & C\int_{Y_1}\left(d(x)^2|\nabla w|^2+d(x)^4|\nabla q|^2+d(x)^2|q|^2\right)dx \\
\leq\ & C(at)^{12}\int_{Y_1}|w|^2dx.
\end{aligned} \tag{4.7}$$

On the other hand, let $X = Z_1 := \{x : \frac{1}{2a} \leq |x| \leq \frac{1}{a} - \frac{1}{at}\}$, then

$$d(x) \geq Ct^{-1}|x| \quad \text{for all} \quad x \in Z,$$

where C another absolute constant. Thus, it follows from (3.1) that

$$\begin{aligned}
&\int_Z \left(|x|^2|\nabla w|^2 + |x|^4|\nabla q|^2 + |x|^2|q|^2\right) dx \\
\leq\ & C(at)^4 \int_{Z_1} \left(d(x)^2|\nabla w|^2 + d(x)^4|\nabla q|^2 dx + d(x)^2|q|^2\right) dx \\
\leq\ & C(at)^{16} \int_{Z_1} |w|^2 dx. \qquad (4.8)
\end{aligned}$$

Combining (4.6), (4.7), and (4.8) implies that

$$\begin{aligned}
& b_1^{-2\beta-n}(\log b_1)^{-4\beta-2} \int_{\frac{1}{at}<|x|<b_1} |w|^2 dx \\
\leq\ & C(at)^{16} b_2^{-2\beta-n}(\log b_2)^{-4\beta+4} \int_{Y_1} |w|^2 dx \\
& + C(at)^{20} b_3^{-2\beta-n}(\log b_3)^{-4\beta+4} \int_{Z_1} |w|^2 dx. \qquad (4.9)
\end{aligned}$$

Replacing $2\beta + n$ by β, (4.9) becomes

$$\begin{aligned}
& b_1^{-\beta}(\log b_1)^{-2\beta+2n-2} \int_{\frac{1}{at}<|x|<b_1} |w|^2 dx \\
\leq\ & C(at)^{16} b_2^{-\beta}(\log b_2)^{-2\beta+2n+4} \int_{Y_1} |w|^2 dx \\
& + C(at)^{20} b_3^{-\beta}(\log b_3)^{-2\beta+2n+4} \int_{Z_1} |w|^2 dx. \qquad (4.10)
\end{aligned}$$

Dividing $b_1^{-\beta}(\log b_1)^{-2\beta+2n-2}$ on the both sides of (4.10) and noting that $\beta \geq$

$n+2>n-1$, i.e., $2\beta-2n+2>0$, we get

$$
\begin{aligned}
&\int_{|x+\frac{b_4x_0}{t}|<\frac{1}{at}}|w(x)|^2dx\\
\leq\;&\int_{\frac{1}{at}<|x|<b_1}|w(x)|^2dx\\
\leq\;&C(at)^{16}(\log(4at))^6(b_1/b_2)^\beta\int_{Y_1}|w|^2dx\\
&+C(at)^{20}(b_1/b_3)^\beta[\log b_1/\log b_3]^{2\beta-2n+4}\int_{Z_1}|w|^2dx\\
\leq\;&C(at)^{16}(\log(4at))^6(4t)^\beta\int_{|x|<\frac{1}{at}}|w(x)|^2dx+C(at)^{20}(b_1/b_5)^\beta\int_{Z_1}|w(x)|^2dx,
\end{aligned}
\tag{4.11}
$$

where $b_4=\frac{1}{a}-\frac{10}{at}$ and $b_5=\frac{1}{a}-\frac{6}{at}$. In deriving the third inequality above, we use the fact that

$$(\frac{b_5}{b_3})(\frac{\log b_1}{\log b_3})^2\leq 1$$

for all $t\geq t_0'$ and $s\leq\tilde{R}_2$, where t_0' and $\tilde{R}_2$ are absolute constants. So we pick $s<\min\{8r_1,8\tilde{R}_1,\tilde{R}_2\}$ and fix it from now on. We observe that s depends only on n.

From (4.11), (2.2) and the definition of $w(x)$, the change of variables $y=atx+x_0$ leads to

$$
\begin{aligned}
M(10)\;&\leq\;Ct^{16}(\log(4at))^6(4t)^\beta\int_{|y-x_0|<1}|u(y)|^2dy+C(\lambda^2\omega_n)t^{20+n}\\
&\leq\;C(4t)^{\beta+22}\int_{|y-x_0|<1}|u(y)|^2dy+Ct^{20+n}(\frac{t}{t+2})^\beta\\
&\leq\;C(4t)^{2\beta}\int_{|y-x_0|<1}|u(y)|^2dy+Ct^{20+n}(\frac{t}{t+2})^\beta,
\end{aligned}
\tag{4.12}
$$

where ω_n is the volume of the unit ball and thus C depends on λ, n. It should be noted that (4.12) holds for all $t\geq t_0''$, $\beta\geq\tilde{\beta}(\geq 22)$, where t_0'' depends only on n. For simplicity, by denoting

$$A(t)=2\log 4t,\quad B(t)=\log(\frac{t+2}{t}),$$

(4.12) becomes

$$M(10) \leq C\Big\{ \exp(\beta A(t)) \int_{|y-x_0|<1} |u(y)|^2 dy + t^{20+n} \exp(-\beta B(t))\Big\}. \quad (4.13)$$

Now, we consider two cases. If

$$\exp(\tilde{\beta} A(t)) \int_{|y-x_0|<1} |u(y)|^2 dy \geq t^{20+n} \exp(-\tilde{\beta} B(t)),$$

then we have

$$\begin{aligned} \int_{|y-x_0|<1} |u(y)|^2 dy &\geq t^{20+n} \exp(-\tilde{\beta} A(t) - \tilde{\beta} B(t)) \\ &\geq t^{20+n} \left(\frac{t+2}{t}\right)^{-\tilde{\beta}} (4t)^{-2\tilde{\beta}} \\ &\geq (4t)^{-3\tilde{\beta}} \geq \exp(-Ct^2 \log t), \end{aligned} \quad (4.14)$$

where C depends on λ and n and $t \geq t_0'''$.

On the other hand, if

$$\exp(\tilde{\beta} A(t)) \int_{|y-x_0|<1} |u(y)|^2 dy \leq t^{20+n} \exp(-\tilde{\beta} B(t)),$$

then we can pick a $\beta > \tilde{\beta}$ such that

$$\exp(\beta A(t)) \int_{|y-x_0|<1} |u(y)|^2 dy = t^{20+n} \exp(-\beta B(t)).$$

Using such β, we obtain from (4.13) that

$$\begin{aligned} M(10) &\leq C \exp(\beta A(t)) \int_{|y-x_0|<1} |u(y)|^2 dy \\ &= C \left(\int_{|y-x_0|<1} |u(y)|^2 dy\right)^{\tau} (t^{20+n})^{1-\tau}, \end{aligned} \quad (4.15)$$

where $\tau = \frac{B(t)}{A(t)+B(t)}$. Thus, (4.15) implies that

$$t^{20+n} \leq \left(\int_{|y-x_0|<1} |u(y)|^2 dy\right) \left(\frac{t^{20+n} C}{M(10)}\right)^{1/\tau}. \quad (4.16)$$

In view of the formula for τ, we can see that

$$\frac{1}{\tau}\log\left(\frac{t^{20+n}C}{M(10)}\right)=\frac{2\log(4t)+\log(1+(2/t))}{\log(1+(2/t))}\log\left(\frac{t^{20+n}C}{M(10)}\right)\leq\tilde{\beta}$$

for all $t>\tilde{t}$. It suffices to choose $\tilde{t}\geq\max\{t_0',t_0'',t_0'''\}$. It is obvious that $\tilde{t}$ depends on λ, n, and $M(10)$. Therefore, we get from (4.16) that

$$\int_{|y-x_0|<1}|u(y)|^2dy\geq t^{20+n}\exp(-Ct^2), \tag{4.17}$$

where C depends on λ and n. Theorem 1.1 now follows from (4.14) and (4.17). □

Acknowledgements

Lin and Wang were supported in part by the National Science Council of Taiwan. Uhlmann was partially supported by NSF, a Chancellor Professorship at UC Berkeley and a Senior Clay Award.

References

[1] J. Bourgain and C. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math. **161** (2005), 1389-426.

[2] R.H. Dyer and D.E. Edmunds, *Asymptotic behavior of solutions of the stationary Navier-Stokes equations*, J. London Math. Soc., **44** (1969), 340-346.

[3] R. Finn, *Stationary solutions of the Navier-Stokes equations*, Proc. Symp. Appl. Math. Amer. Math. Soc, **17** (1965), 121-153.

[4] L. Hörmander, "The analysis of linear partial differential operators", Vol. 3, Springer-Verlag, Berlin/New York, 1985.

[5] C. Kenig, *Lecture Notes for 2006 CNA Summer School: Probabilistic and Analytical Perspectives on Contemporary PDEs*, Center for Nonlinear Analysis, Carnegie Mellon University.

[6] C.L. Lin, G. Uhlmann and J.N. Wang, *Optimal three-ball inequalities and quantitative uniqueness for the Stokes system*, to appear in DSDS-A, A special issue dedicated to Professor Louis Nirenberg on the occasion of his 85th birthday.

[7] C.L. Lin, G. Nakamura and J.N. Wang, *Optimal three-ball inequalities and quantitative uniqueness for the Lamé system with Lipschitz coefficients*, to appear in Duke Math Journal.

[8] M. Mitrea and S. Monniaux, *Maximal regularity for the Lamé system in certain classes of non-smooth domains*, J. Evol. Equ., DOI 10.1007/s00028-010-0071-1.