Determination of second-order elliptic operators in two dimensions from partial Cauchy data

Oleg Yu. Imanuvilov [*], **Gunther Uhlmann** [†], **and Masahiro Yamamoto** [‡]

[*]Department of Mathematics, Colorado State University, Fort Collins CO, 80523 USA, e-mail: oleg@math.colostate.edu,[†]Department of Mathematics, University of California Irvine, Irvine, CA 92697-3875 and Department of Mathematics University of Washington, Seattle, WA 98195 USA, e-mail: gunther@math.washington.edu, and [‡]Department of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo 153-8914, Japan, e-mail: myama@ms.u-tokyo.ac.jp

Submitted to Proceedings of the National Academy of Sciences of the United States of America

We consider the inverse boundary value problem in two dimensions of determining the coefficients of a general second-order elliptic operator from the Cauchy data measured on a non-empty arbitrary relatively open subset of the boundary. We give a complete characterization of the set of coefficients yielding the same partial Cauchy data. As a corollary we prove several uniqueness results in determining coefficients from partial Cauchy data for the isotropic conductivity equation, the so-called Calderón's problem [5], the Schrödinger equation, the convection-diffusion equation, the anisotropic conductivity equation modulo a group of diffeomorphisms that are the identity at the boundary, and the magnetic Schrödinger equations modulo gauge transformations. The key step is the construction of novel complex geometrical optics solutions using Carleman estimates.

partial Cauchy data, general second-order elliptic equations, complex geometrical optics solutions

1 Main result

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary $\partial\Omega = \cup_{k=1}^{\mathcal{N}} \gamma_k$, where γ_k, $1 \le k \le \mathcal{N}$, are smooth closed contours, and $\gamma_{\mathcal{N}}$ is the external contour. Let $\widetilde{\Gamma} \subset \partial\Omega$ be an arbitrarily fixed non-empty relatively open subset of $\partial\Omega$. Let ν be the unit outward normal vector to $\partial\Omega$ and let $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$. We set $i = \sqrt{-1}$ and identify $x = (x_1, x_2) \in \mathbf{R}^2$ with $z = x_1 + ix_2 \in \mathbf{C}$, and by $\overline{z}$ we denote the complex conjugate of $z \in \mathbf{C}$.

We consider a second-order elliptic operator:

$$L(x,D)u = \Delta_g u + 2A\frac{\partial u}{\partial z} + 2B\frac{\partial u}{\partial \overline{z}} + qu. \qquad [1]$$

Here $g = g(x) = \{g_{jk}\}_{1\le j,k\le 2}$ is a positive definite symmetric matrix in Ω and Δ_g is the Laplace-Beltrami operator associated to the Riemannian metric g:

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{2} \frac{\partial}{\partial x_k}(\sqrt{\det g}\, g^{jk} \frac{\partial}{\partial x_j}),$$

where we set $\{g^{jk}\} = g^{-1}$. Throughout this paper, we assume that $g \in C^{7+\alpha}(\overline{\Omega})$, $(A,B,q), (A_j,B_j,q_j) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \times C^{4+\alpha}(\overline{\Omega})$, $j = 1,2$ for some $\alpha \in (0,1)$, are complex-valued functions. We set

$$L_k(x,D) = \Delta_{g_k} + 2A_k\frac{\partial}{\partial z} + 2B_k\frac{\partial}{\partial \overline{z}} + q_k.$$

We define the set of partial Cauchy data by

$$\mathcal{C}_{g,A,B,q} = \{(u|_{\widetilde{\Gamma}}, \frac{\partial u}{\partial \nu_g}|_{\widetilde{\Gamma}});$$
$$L(x,D)u = 0 \text{ in } \Omega, u \in H^1(\Omega),\, u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0\},$$

where $\frac{\partial}{\partial\nu_g} = \sqrt{\det g}\sum_{j,k=1}^{2} g^{jk}\nu_k \frac{\partial}{\partial x_j}$ is the conormal derivative with respect to the metric g.

The goal of this paper is to determine the metric g and coefficients A, B, q from the partial Cauchy data $\mathcal{C}_{g,A,B,q}$. In the general case this is impossible. There are the following main invariants of the Cauchy data in the problem.

- *Conformal Invariance.* Let $\beta \in C^{7+\alpha}(\overline{\Omega})$ be a strictly positive function. Then

$$\mathcal{C}_{g,A,B,q} = \mathcal{C}_{\beta g, \frac{A}{\beta}, \frac{B}{\beta}, \frac{q}{\beta}}. \qquad [2]$$

  This follows since the Laplace-Beltrami operator is conformal invariant in two dimensions:

$$\Delta_{\beta g} = \frac{1}{\beta}\Delta_g.$$

- *Gauge Transformations.* It is easy to see that the set of partial Cauchy data of the operators $e^{-\eta}L(x,D)e^{\eta}$ and $L(x,D)$ are the same provided that η is a smooth complex-valued function such that

$$\eta \in C^{6+\alpha}(\bar\Omega), \quad \eta|_{\widetilde{\Gamma}} = \frac{\partial\eta}{\partial\nu}|_{\widetilde{\Gamma}} = 0. \qquad [3]$$

- *Diffeomorphism Invariance.* Let $F = (F_1, F_2) : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism such that $F|_{\widetilde{\Gamma}} = Id$. The pull back of a Riemannian metric g is given as composition of matrices by

$$F^*g = ((DF) \circ g \circ (DF)^T) \circ F^{-1} \qquad [4]$$

  where DF denotes the differential of F, $(DF)^T$ its transpose and $\circ$ denotes matrix composition.
  Moreover we introduce the functions: $A_F = \{(A+B)(\frac{\partial F_1}{\partial x_1} - i\frac{\partial F_2}{\partial x_1}) + i(B-A)(\frac{\partial F_1}{\partial x_2} - i\frac{\partial F_2}{\partial x_2})\}\circ F^{-1}|det\, DF^{-1}|$, $B_F = \{(A+B)(\frac{\partial F_1}{\partial x_1} + i\frac{\partial F_2}{\partial x_1}) + i(B-A)(\frac{\partial F_1}{\partial x_2} + i\frac{\partial F_2}{\partial x_2})\} \circ F^{-1}|det\, DF^{-1}|$, $q_F = |det\, DF^{-1}|(q \circ F^{-1})$. Then

$$\mathcal{C}_{g,A,B,q} = \mathcal{C}_{F^*g, A_F, B_F, q_F}. \qquad [5]$$

We show the converse, namely, a complete list of invariants of the problem. We have

Theorem 1. *Suppose that for some $\alpha \in (0,1)$, there exists a positive function $\widetilde{\beta} \in C^{7+\alpha}(\overline{\Omega})$ such that $(g_1 - \widetilde{\beta}g_2)|_{\widetilde{\Gamma}} = \frac{\partial(g_1 - \widetilde{\beta}g_2)}{\partial\nu}|_{\widetilde{\Gamma}} = 0$. Then $\mathcal{C}_{g_1,A_1,B_1,q_1} = \mathcal{C}_{g_2,A_2,B_2,q_2}$ if and only if there exist a diffeomorphism $F \in C^{8+\alpha}(\overline{\Omega})$, $F : \overline{\Omega} \to \overline{\Omega}$ satisfying $F|_{\widetilde{\Gamma}}=Id$, a positive function $\beta \in C^{7+\alpha}(\overline{\Omega})$ and a complex valued function η satisfying (3) such that*

$$L_2(x,D) = e^{-\eta}K(x,D)e^{\eta},$$

Reserved for Publication Footnotes

where

$$K(x,D)=\Delta_{\beta F^*g_1}+\frac{2}{\beta}(A_{1F}\frac{\partial}{\partial z}+B_{1F}\frac{\partial}{\partial \bar z})+\frac{1}{\beta}q_{1F}.$$

2 Calderón's problem and other applications

2.1 Calderón's Problem. The question proposed by Calderón [5] is whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary.

In the anisotropic case the conductivity depends on direction and is represented by a positive definite symmetric matrix $\{\sigma^{jk}\}$. The conductivity equation with voltage potential f on $\partial\Omega$ is given by

$$\sum_{j,k=1}^{2}\frac{\partial}{\partial x_j}(\sigma^{jk}\frac{\partial u}{\partial x_k})=0\quad \text{in }\Omega,\quad u|_{\partial\Omega}=f\in H^{\frac12}(\partial\Omega).$$

We define the partial Cauchy data by

$$\mathcal{V}_\sigma=\left\{\left(f|_{\tilde\Gamma},\sum_{j,k=1}^{2}\sigma^{jk}\nu_j\frac{\partial u}{\partial x_k}\Big|_{\tilde\Gamma}\right)\Big|\sum_{j,k=1}^{2}\frac{\partial}{\partial x_j}(\sigma^{jk}\frac{\partial u}{\partial x_k})=0\right. \quad [6]$$

$$\left.\text{in }\Omega,\ \ u\in H^1(\Omega),\ u|_{\partial\Omega}=f,\,\text{supp } f\subset\widetilde\Gamma\right\}.$$

It has been known for a long time that $\mathcal{V}_\sigma$ does not determine σ uniquely in the anisotropic case [10]. Let $F:\overline\Omega\to\overline\Omega$ be a diffeomorphism such that $F(x)=x$ for x on $\tilde\Gamma$. Then

$$\mathcal{V}_{|det\,DF^{-1}|F^*\sigma}=\mathcal{V}_\sigma,$$

where $F^*\sigma$ is given by (4).

In the case of full Cauchy data (i.e., $\widetilde\Gamma=\partial\Omega$), the question whether one can determine the conductivity up to the above obstruction has been solved in two dimensions for C^2 conductivities in [11], Lipschitz conductivities in [13] and merely L^∞ conductivities in [3]. See also [2]. The method of proof in all these papers is based on the reduction to the isotropic case using isothermal coordinates [1].

We can prove the uniqueness for Calderón's problem with partial Cauchy data:

Theorem 2. *Let $\sigma_1,\sigma_2\in C^{7+\alpha}(\overline\Omega)$ with some $\alpha\in(0,1)$ be positive definite symmetric matrices on $\overline\Omega$. If $\mathcal{V}_{\sigma_1}=\mathcal{V}_{\sigma_2}$ then there exists a diffeomorphism $F:\overline\Omega\to\overline\Omega$ satisfying $F|_{\widetilde\Gamma}=Id$ and $F\in C^{8+\alpha}(\overline\Omega)$ such that*

$$|det\,DF^{-1}|F^*\sigma_1=\sigma_2.$$

For the isotropic case this result was proven in [8] and in fact follows from Theorem 1 in the case where $g=I$ and $A=B=0$. We mention that [7] has proven a similar result for general Riemann surfaces in the case where g is not the identity but fixed.

2.2 Case where the principal part is the Laplacian. In the rest of section 2, we assume that the principal parts of second order elliptic operators under consideration are the Laplacian: $g=I\equiv\{\delta_{jk}\}$.

For the case when $A_1=A_2,B_1=B_2$ and full data this result was proven by Bukgheim [4].

Theorem 3. *If $\mathcal{C}_{I,A_1,B_1,q_1}=\mathcal{C}_{I,A_2,B_2,q_2}$, then*

$$A_1=A_2,\quad B_1=B_2\quad on\quad \widetilde\Gamma, \quad [7]$$

and in the domain Ω we have

$$-2\frac{\partial}{\partial z}(A_1-A_2)-A_1B_1+A_2B_2+(q_1-q_2)=0, \quad [8]$$

$$-2\frac{\partial}{\partial \bar z}(B_1-B_2)-A_1B_1+A_2B_2+(q_1-q_2)=0. \quad [9]$$

Corollary 4. *The relation $\mathcal{C}_{I,A_1,B_1,q_1}=\mathcal{C}_{I,A_2,B_2,q_2}$ holds true if and only if there exists a function $\eta\in C^{6+\alpha}(\overline\Omega)$ satisfying $\eta|_{\tilde\Gamma}=\frac{\partial\eta}{\partial\nu}|_{\tilde\Gamma}=0$ such that*

$$L_1(x,D)=e^{-\eta}L_2(x,D)e^{\eta}. \quad [10]$$

Proof of Corollary 4. We only prove the sufficiency since the necessity of the condition is easy to check. By (8) and (9), we have $\frac{\partial}{\partial z}(A_1-A_2)=\frac{\partial}{\partial\bar z}(B_1-B_2)$. This equality is equivalent to

$$\frac{\partial(\widehat A-\widehat B)}{\partial x_1}=i\frac{\partial(\widehat B+\widehat A)}{\partial x_2}\quad \text{where}\quad (\widehat A,\widehat B)=(A_1-A_2,B_1-B_2).$$

Applying Lemma 1.1 (p.313) of [14], we obtain that there exists a function $\tilde\eta$ in the domain Ω^0 which satisfies

$$\tilde\eta=\eta_0+h,\nabla\tilde\eta\in C^{5+\alpha}(\overline\Omega),\ \Delta h=0\quad\text{in }\Omega^0, \quad [11]$$

$$[h]|_{\Sigma_k}\ \text{are constants,}\ \left[\frac{\partial h}{\partial\nu_k}\right]|_{\Sigma_k}=\frac{\partial h}{\partial\nu}|_{\gamma_{\mathcal N}}=0\quad\forall k\in\{1,\dots,\mathcal N\}$$

and

$$(i(\widehat B+\widehat A),(\widehat A-\widehat B))=\nabla\tilde\eta.$$

Here $\Omega^0=\Omega\setminus\Sigma$ is simply connected where $\Sigma=\cup_{k=1}^{\mathcal N-1}\Sigma_k$, $\Sigma_j\cap\Sigma_k=\emptyset$ for $j\ne k$, Σ_k are smooth curves which do not self-intersect and are orthogonal to $\partial\Omega$. We choose a normal vector $\nu_k=\nu_k(x)$, $1\le k\le\mathcal N-1$ to Σ_k at x contained in the interior Σ_k^0 of the closed curve Σ_k. Then, for $x\in\Sigma_k^0$, we set $[h](x)=\lim_{y\to x,(\overrightarrow{xy},\nu_k)>0}h(y)-\lim_{y\to x,(\overrightarrow{xy},\nu_k)<0}h(y)$ where $(\cdot,\cdot)$ denotes the scalar product in $\mathbf{R}^2$. Setting $2\eta=-i\tilde\eta$ we have

$$((\widehat B+\widehat A),i(\widehat B-\widehat A))=2\nabla\eta.$$

Therefore by (8)

$$q_1=q_2+\Delta\eta+4\frac{\partial\eta}{\partial z}\frac{\partial\eta}{\partial\bar z}+2\frac{\partial\eta}{\partial z}A_2+2\frac{\partial\eta}{\partial\bar z}B_2. \quad [12]$$

The operator $L_1(x,D)$ given by the right hand side of (10) has the Laplace operator as the principal part, the coefficients of $\frac{\partial}{\partial x_1}$ is $A_2+B_2+2\frac{\partial\eta}{\partial x_2}$, the coefficient of $\frac{\partial}{\partial x_2}$ is $i(B_2-A_2)+2\frac{\partial\eta}{\partial x_1}$, and the coefficient of the zero order term is given by the right-hand side of (12). By (7) we have that $\frac{\partial\eta}{\partial\nu}|_{\tilde\Gamma}=0$ and $\eta|_{\tilde\Gamma}=\mathcal{C}$ where the function $\mathcal{C}(x)$ is equal to a constant on each connected component of $\tilde\Gamma$. Let us show that the function η is continuous. Our proof is by contradiction. Suppose that η is discontinuous say along the curve Σ_j. Let the function $u_2\in H^1(\Omega)$ be a solution to the following boundary value problem

$$L_2(x,D)u_2=0\quad\text{in }\Omega,\quad u_2|_{\Gamma_0}=0. \quad [13]$$

Assume in addition that u_2 is not identically equal to zero on Σ_j. Let $\tilde\Gamma_1$ be one connected component of the set $\tilde\Gamma$ and $\mathcal{C}|_{\tilde\Gamma_1}=\hat C$. Without loss of generality, we may assume that $\hat C=0$. Indeed if $\hat C\ne0$ we replace η by the function $\eta-\hat C$. Since the partial Cauchy data generated by the operators $L_1(x,D)$ and $L_2(x,D)$ are the same, there exists a solution u_1 to the following boundary value problem

$$L_1(x,D)u_1=0\quad\text{in }\Omega,\quad u_1=u_2\quad\text{on }\partial\Omega,\quad \frac{\partial u_1}{\partial\nu}=\frac{\partial u_2}{\partial\nu}\quad\text{on }\tilde\Gamma. \quad [14]$$

Then the function $v = e^{-\eta}u_2$ verifies

$$L_1(x, D)v = 0 \quad \text{in } \Omega^0, \quad v|_{\Gamma_0} = 0.$$

Since $\eta = \frac{\partial \eta}{\partial \nu} = 0$ on $\tilde{\Gamma}_1$, we have that $v \equiv u_1$. On the other hand, $u_1 \in H^1(\Omega)$ and v are discontinuous along one part of Σ_j, and we arrive at a contradiction.

Let us show that $\mathcal{C} \equiv 0$. Suppose that there exists another connected component of $\tilde{\Gamma}_2$ of the set $\tilde{\Gamma}$ such that $\mathcal{C}|_{\tilde{\Gamma}_2} \neq 0$. Assume that u_1, u_2 satisfy (13), (14).

Then the function $v = e^{-\eta}u_2$ verifies

$$L_1(x, D)v = 0 \quad \text{in } \Omega, \quad v|_{\Gamma_0} = 0.$$

Moreover, since $\eta = \frac{\partial \eta}{\partial \nu} = 0$ on $\tilde{\Gamma}_1$, we have that

$$v = u_1, \quad \frac{\partial v}{\partial \nu} = \frac{\partial u_1}{\partial \nu} \quad \text{on} \quad \tilde{\Gamma}_1.$$

The uniqueness of the Cauchy problem for the second-order elliptic equation yields $v \equiv u_1$. In particular $v = u_1$ on $\tilde{\Gamma}_2$. Since $u_1 = u_2$ on $\partial\Omega$, this implies that $e^{-\eta}|_{\tilde{\Gamma}_2} = 1$. We arrived at a contradiction. The proof of the corollary is completed. $\square$

Next we apply Theorem 3 to several cases and state new results on the unique identifiability, modulo the natural obstructions, of some important inverse boundary value problems with partial Cauchy data arising in Mathematical Physics .

2.3 The magnetic Schrödinger equation. We consider the case of the magnetic Schrödinger operator.

Denote $\widetilde{A} = (\widetilde{A}_1, \widetilde{A}_2)$, where $\widetilde{A}_j$ are real-valued, $\widetilde{\mathcal{A}} = \widetilde{A}_1 - i\widetilde{A}_2$, rot $\widetilde{A} = \frac{\partial \widetilde{A}_2}{\partial x_1} - \frac{\partial \widetilde{A}_1}{\partial x_2}$. The magnetic Schrödinger operator is defined by

$$\mathcal{L}_{\widetilde{A},\widetilde{q}}(x, D) = \sum_{k=1}^{2} (\frac{1}{i}\frac{\partial}{\partial x_k} + \widetilde{A}_k)^2 + \widetilde{q}.$$

Let us define the following set of partial Cauchy data

$$\widetilde{C}_{\widetilde{A},\widetilde{q}} = \{(u|_{\tilde{\Gamma}}, \frac{\partial u}{\partial \nu}|_{\tilde{\Gamma}}); \mathcal{L}_{\widetilde{A},\widetilde{q}}(x, D)u = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega\setminus\tilde{\Gamma}} = 0, \, u \in H^1(\Omega)\}.$$

Theorem 3 implies

Corollary 5. *Let real-valued vector fields $\widetilde{A}^{(1)}, \widetilde{A}^{(2)} \in C^{5+\alpha}(\overline{\Omega})$ and complex-valued potentials $\widetilde{q}^{(1)}, \widetilde{q}^{(2)} \in C^{4+\alpha}(\overline{\Omega})$ with some $\alpha \in (0, 1)$, satisfy $\widetilde{C}_{\widetilde{A}^{(1)},\widetilde{q}^{(1)}} = \widetilde{C}_{\widetilde{A}^{(2)},\widetilde{q}^{(2)}}$. Then $\widetilde{q}^{(1)} = \widetilde{q}^{(2)}$, $rot\,\widetilde{A}^{(1)} = rot\,\widetilde{A}^{(2)}$ and $\widetilde{A}^{(1)} = \widetilde{A}^{(2)}$ on $\tilde{\Gamma}$.*

We mention that this result is new even for the case of full data. In this case, [12] proved a uniqueness result assuming that both the electric and magnetic potentials are small. Still in the case of full data, [9] proved a uniqueness result for a special case of the magnetic Schrödinger equation, namely the Pauli Hamiltonian.

2.4 Laplace equation with convection terms. Another application of Theorem 3 is to the Laplace equation with convection terms. For real-valued a, b, and complex valued q, we define the following set of partial Cauchy data

$$\widetilde{C}_{a,b,q} = \{(u|_{\tilde{\Gamma}}, \frac{\partial u}{\partial \nu}|_{\tilde{\Gamma}}); u|_{\partial\Omega\setminus\tilde{\Gamma}} = 0, u \in H^1(\Omega),$$
$$\Delta u + a\frac{\partial u}{\partial x_1} + b\frac{\partial u}{\partial x_2} + qu = 0 \text{ in } \Omega\}.$$

Then

Corollary 6. *Let $\alpha \in (0, 1)$, $q \in C^{4+\alpha}(\overline{\Omega})$, and $(a^{(j)}, b^{(j)}) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega})$. If $\widetilde{C}_{a^{(1)},b^{(1)},q} = \widetilde{C}_{a^{(2)},b^{(2)},q}$, then $(a^{(1)}, b^{(1)}) \equiv (a^{(2)}, b^{(2)})$.*

This corollary generalizes the result of [6] where the uniqueness was proved assuming that the measurements are made on the whole boundary.

We also mention that Theorem 3 implies that partial Cauchy data on arbitrary $\widetilde{\Gamma}$ uniquely determine any two coefficients of the triple (A, B, q). A particular case is:

Corollary 7. *For $j = 1, 2$, let $(A_j, B_j, q_j) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega}) \times C^{4+\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ be complex-valued. We assume either $A_1 = A_2$ or $B_1 = B_2$ in Ω. Then $\mathcal{C}_{I,A_1,B_1,q_1} = \mathcal{C}_{I,A_2,B_2,q_2}$ implies $(A_1, B_1, q_1) = (A_2, B_2, q_2)$.*

3 Sketch of Proof of Theorem 3

The key of the proof is the constructions of families of τ-parameterized solutions $u_1 = u_1(\tau)(x)$ and $v = v(\tau)(x)$ with $\tau \in \mathbf{R}$ satisfying $L_1(x, D)u_1 = 0$, $u_1|_{\Gamma_0} = 0$ and $L_2(x, D)^*v = 0$, $v|_{\Gamma_0} = 0$. Here $L_2(x, D)^*$ is the adjoint to $L_2(x, D)$ and $\Gamma_0 = \partial\Omega \setminus \widetilde{\Gamma}$. By u_2 we denote the solution to $L_2(x, D)u_2 = 0$ with $u_2|_{\partial\Omega} = u_1|_{\partial\Omega}$. Then the coincidence of the partial Cauchy data yields $\nabla u_1(\tau) = \nabla u_2(\tau)$ on $\widetilde{\Gamma}$. Therefore integration by parts gives

$$0 = \int_\Omega \bar{v}L_2(x, D)(u_1 - u_2)dx = \int_\Omega (2(A_1 - A_2)\frac{\partial u_1}{\partial z}$$
$$+2(B_1 - B_2)\frac{\partial u_1}{\partial \overline{z}} + (q_1 - q_2)u_1)\bar{v}dx. \quad \textbf{[15]}$$

Then the proof relies on the constructions of suitable u_1 and v which are complex geometrical optics solutions.

Complex geometrical optics (CGO) solution. We look for the geometrical optics solution u_1 of the form:

$$u_1(x) = a_\tau(z)e^{\mathcal{A}_1+\tau\Phi} + d_\tau(\overline{z})e^{\mathcal{B}_1+\tau\overline{\Phi}} + u_{11}e^{\tau\varphi} + u_{12}e^{\tau\varphi}. \quad \textbf{[16]}$$

The phase function. Let the holomorphic function $\Phi = \varphi + i\psi$ satisfy

$$\text{Im}\,\Phi|_{\Gamma_0} = 0, \; \mathcal{H} \cap \partial\Omega \subset \Gamma_0, \quad \frac{\partial^2 \Phi}{\partial z^2}(z) \neq 0, \, \forall z \in \mathcal{H}, \quad \textbf{[17]}$$

where $\mathcal{H} = \{z \in \overline{\Omega}; \, \frac{\partial \Phi}{\partial z} = 0\}$. The critical points of the function Φ play important role in the proof. The proposition below shows that the union of the sets of critical points of the functions satisfying (17) is dense in Ω.

Proposition 8. *Let $\widetilde{x}$ be an arbitrary point in Ω. There exists a sequence of functions $\{\Phi_\epsilon\}_{\epsilon\in(0,1)}$ satisfying (17) such that there exists a sequence $\{\widetilde{x}_\epsilon\}, \epsilon \in (0, 1)$ and*

$$\widetilde{x}_\epsilon \in \mathcal{H}_\epsilon = \{z \in \overline{\Omega} | \frac{\partial \Phi_\epsilon}{\partial z}(z) = 0\}, \quad \widetilde{x}_\epsilon \to \widetilde{x} \;\; \textit{as } \epsilon \to +0.$$

Moreover for any j from $\{1, \dots, \mathcal{N}\}$ we have

$$\mathcal{H}_\epsilon \cap \gamma_j = \emptyset \quad \textit{if } \gamma_j \cap \tilde{\Gamma} \neq \emptyset,$$

$$\mathcal{H}_\epsilon \cap \gamma_j \subset \Gamma_0 \quad \textit{if } \gamma_j \cap \tilde{\Gamma} = \emptyset,$$

$$Im\,\Phi_\epsilon(\widetilde{x}_\epsilon) \notin \{Im\,\Phi_\epsilon(x) | x \in \mathcal{H}_\epsilon \setminus \{\widetilde{x}_\epsilon\}\} \textit{ and } Im\,\Phi_\epsilon(\widetilde{x}_\epsilon) \neq 0.$$

Let $\vec{\tau}$ be a tangential vector field to $\partial\Omega$. In order to prove (7), we use the phase function Φ given by the following lemma.

Proposition 9. *Let $\Gamma_* \subset\subset \widetilde{\Gamma}$ be an arc oriented clockwise with left endpoint x_- and right endpoint x_+. For any $\widehat{x} \in Int\,\Gamma_*$ there exists a function $\Phi(z)$ which satisfies (17), $Im\,\Phi|_{\partial\Omega\setminus\Gamma_*} = 0$ and*

$$\widehat{x} \in \mathcal{G} = \{x \in \Gamma_* \,|\quad \frac{\partial Im\,\Phi}{\partial\vec{\tau}}(x) = 0\}, \quad card\,\mathcal{G} < \infty,$$

all the critical points of $Im\,\Phi$ from the set $\mathcal{G}\setminus\{x_-,x_+\}$ are nondegenerate, and the left or the right derivative of $Im\,\Phi$ of order seven is not equal to zero at $x_\pm$.

The functions $\mathcal{A}_1, \mathcal{B}_1 \in C^{6+\alpha}(\overline{\Omega})$ are defined by $2\frac{\partial\mathcal{A}_1}{\partial\overline{z}} = -A_1$ in Ω, $\mathrm{Im}\,\mathcal{A}_1|_{\Gamma_0} = 0$, $2\frac{\partial\mathcal{B}_1}{\partial z} = -B_1$ in Ω, $\mathrm{Im}\,\mathcal{B}_1|_{\Gamma_0} = 0$. The amplitudes are of the forms $a_\tau(z) = a(z) + \frac{a_1(z)}{\tau} + \frac{a_{2,\tau}(z)}{\tau^2}$, $d_\tau(\overline{z}) = d(\overline{z}) + \frac{d_1(\overline{z})}{\tau} + \frac{d_{2,\tau}(\overline{z})}{\tau^2}$, where a is a holomorphic function and d is an antiholomorpic function such that $a(z)e^{\mathcal{A}_1} + d(\overline{z})e^{\mathcal{B}_1} = 0$ on Γ_0. Here and henceforth, if $\partial_z a(z) = 0$, then we call a antiholomorphic.

Let $\tilde{x}$ be some fixed point from $\mathcal{H}\setminus\partial\Omega$. In addition the functions a and d have the following properties

$$\frac{\partial^k a}{\partial z^k}|_{\mathcal{H}\cap\partial\Omega} = 0, \quad \frac{\partial^k d}{\partial\overline{z}^k}|_{\mathcal{H}\cap\partial\Omega} = 0 \quad \forall k \in \{0,\dots,5\},$$
$$a|_{\mathcal{H}\setminus\{\tilde{x}\}} = d|_{\mathcal{H}\setminus\{\tilde{x}\}} = 0,\ a(\tilde{x}) \neq 0,\ d(\tilde{x}) \neq 0.$$

We introduce the following operators $T_B g = e^{\mathcal{B}}\partial_z^{-1}(e^{-\mathcal{B}}g)$ and $P_A g = e^{\mathcal{A}}\partial_{\overline{z}}^{-1}(e^{-\mathcal{A}}g)$ and the operators

$$\mathcal{R}_{\tau,A}g = \frac{1}{2}e^{\mathcal{A}}e^{\tau(\overline{\Phi}-\Phi)}\partial_{\overline{z}}^{-1}(ge^{-\mathcal{A}}e^{\tau(\Phi-\overline{\Phi})}),$$
$$\widetilde{\mathcal{R}}_{\tau,B}g = \frac{1}{2}e^{\mathcal{B}}e^{\tau(\overline{\Phi}-\Phi)}\partial_z^{-1}(ge^{-\mathcal{B}}e^{\tau(\Phi-\overline{\Phi})}).$$

Here $2\frac{\partial\mathcal{A}}{\partial\overline{z}} = -A$, $2\frac{\partial\mathcal{B}}{\partial z} = -B$, $\mathrm{Im}\,\mathcal{A}|_{\Gamma_0} = \mathrm{Im}\,\mathcal{B}|_{\Gamma_0} = 0$, $\partial_{\overline{z}}^{-1}g = -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1,\xi_2)}{\zeta-z}d\xi_1 d\xi_2$, $\partial_z^{-1}g = \overline{\partial_{\overline{z}}^{-1}\overline{g}}$, $\zeta = \xi_1 + i\xi_2$.

Denote $g_1 = T_{B_1}((q_1 - 2\frac{\partial B_1}{\partial\overline{z}} - A_1B_1)de^{\mathcal{B}_1}) - M_2(\overline{z})e^{\mathcal{B}_1}$, $g_2 = P_{A_1}((q_1 - 2\frac{\partial A_1}{\partial z} - A_1B_1)ae^{\mathcal{A}_1}) - M_1(z)e^{\mathcal{A}_1}$, where $M_1(z)$ and $M_2(\overline{z})$ are polynomials such that

$$\frac{\partial^k g_1}{\partial\overline{z}^k}|_{\mathcal{H}} = \frac{\partial^k g_2}{\partial z^k}|_{\mathcal{H}} = 0 \quad \forall k \in \{0,\dots,5\}.$$

Thanks to our assumptions on the regularity of A_1, B_1 and q the functions g_1, g_2 belong to $C^{6+\alpha}(\overline{\Omega})$.

The function $a_1(z)$ is holomorphic in Ω and $d_1(\overline{z})$ is antiholomorphic in Ω and

$$a_1(z)e^{\mathcal{A}_1} + d_1(\overline{z})e^{\mathcal{B}_1} = \frac{g_1}{2\overline{\partial_z\Phi}} + \frac{g_2}{2\partial_z\Phi} \quad \text{on}\,\Gamma_0.$$

Construction of the correction term u_{11}. Let

$$\hat{g}_1 = T_{B_1}((q_1 - 2\frac{\partial B_1}{\partial\overline{z}} - A_1B_1)d_1e^{\mathcal{B}_1}) - \hat{M}_2(\overline{z})e^{\mathcal{B}_1},$$

$$\hat{g}_2 = P_{A_1}((q_1 - 2\frac{\partial A_1}{\partial z} - A_1B_1)a_1e^{\mathcal{A}_1}) - \hat{M}_1(z)e^{\mathcal{A}_1},$$

where $\hat{M}_1(z)$ and $\hat{M}_2(\overline{z})$ are polynomials such that

$$\frac{\partial^k \hat{g}_1}{\partial\overline{z}^k}|_{\mathcal{H}} = \frac{\partial^k \hat{g}_2}{\partial z^k}|_{\mathcal{H}} = 0 \quad \forall k \in \{0,\dots,3\}.$$

Let $e_1(x), e_2(x)$ be smooth functions such that $e_1 + e_2 \equiv 1$, e_2 vanishes in some neighborhood of $\mathcal{H}\setminus\Gamma_0$ and e_1 vanishes in some neighborhood of $\partial\Omega$.

The function u_{11} is given by

$$u_{11} = -e^{-i\tau\psi}\mathcal{R}_{-\tau,A_1}\{e_1(g_1 + \hat{g}_1/\tau)\} -$$
$$e^{-i\tau\psi}\frac{e_2(g_1 + \frac{\hat{g}_1}{\tau})}{2\tau\overline{\partial_z\Phi}} + \frac{e^{-i\tau\psi}}{4\tau^2\overline{\partial_z\Phi}}L_1(x,D)\left(\frac{e_2g_1}{\overline{\partial_z\Phi}}\right)$$
$$-e^{i\tau\psi}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1(g_2 + \hat{g}_2/\tau)\}$$
$$-e^{i\tau\psi}\frac{e_2(g_2 + \frac{\hat{g}_2}{\tau})}{2\tau\partial_z\Phi} + \frac{e^{i\tau\psi}}{4\tau^2\partial_z\Phi}L_1(x,D)\left(\frac{e_2g_2}{\partial_z\Phi}\right).$$

Construction of the correction terms $a_{2,\tau}(z)$ and $d_{2,\tau}(\overline{z})$.

Observe that the following asymptotic formulae hold true for any point on the boundary of Ω :

$$\mathcal{R}_{-\tau,A_1}\{e_1g_1\} = \frac{1}{2\tau^2}\frac{e^{2i\tau\psi - 2i\tau\psi(\tilde{x})}p_+}{|\det\psi''(\tilde{x})|^{\frac{1}{2}}} + \mathcal{W}_{\tau,1},$$
$$\tilde{\mathcal{R}}_{\tau,B_1}\{e_1g_2\} = \frac{1}{2\tau^2}\frac{e^{-2i\tau\psi + 2i\tau\psi(\tilde{x})}p_-}{|\det\psi''(\tilde{x})|^{\frac{1}{2}}} + \mathcal{W}_{\tau,2},$$

where $\sigma_1, \tilde{\sigma}_1, m_1, \tilde{m}_1$ are some smooth functions, $p_+(x) = e^{\mathcal{A}_1}\left(\frac{\sigma_1(\tilde{x})}{(z-\tilde{z})^2} + \frac{m_1(\tilde{x})}{(\tilde{z}-z)}\right)$, $p_-(x) = e^{\mathcal{B}_1}\left(\frac{\tilde{\sigma}_1(\tilde{x})}{(\overline{z}-\overline{\tilde{z}})^2} + \frac{\tilde{m}_1(\tilde{x})}{(\overline{\tilde{z}}-\overline{z})}\right)$, $\tilde{z} = \tilde{x}_1 + i\tilde{x}_2$ and $\mathcal{W}_{\tau,1}, \mathcal{W}_{\tau,2}$ satisfy

$$\|\mathcal{W}_{\tau,1}\|_{H^{\frac{1}{2}}(\Gamma_0)} + \|\mathcal{W}_{\tau,2}\|_{H^{\frac{1}{2}}(\Gamma_0)} = o(\frac{1}{\tau^2}) \quad \text{as}\,|\tau| \to +\infty.$$

We define the functions $a_{2,\pm}(z) \in C^2(\overline{\Omega})$ and $d_{2,\pm}(\overline{z}) \in C^2(\overline{\Omega})$ satisfying

$$a_{2,\pm}(z)e^{\mathcal{A}_1} + d_{2,\pm}(\overline{z})e^{\mathcal{B}_1} = p_\pm \quad \text{on}\,\Gamma_0.$$

Let

$$g_5 = \frac{P_{A_1}((q_1 - 2\frac{\partial A_1}{\partial z} - A_1B_1)g_1) - M_5(z)e^{\mathcal{A}_1}}{2\partial_z\Phi},$$
$$g_6 = \frac{T_{B_1}((q_1 - 2\frac{\partial B_1}{\partial\overline{z}} - A_1B_1)g_2) - M_6(\overline{z})e^{\mathcal{B}_1}}{2\overline{\partial_z\Phi}}.$$

Here $M_5(z), M_6(\overline{z})$ are polynomials such that $g_5|_{\mathcal{H}} = g_6|_{\mathcal{H}} = \nabla g_5|_{\mathcal{H}} = \nabla g_6|_{\mathcal{H}} = 0$. Let a holomorphic function $a_{2,0}$ and an antiholomorphic function $d_{2,0}$ satisfy

$$a_{2,0}(z)e^{\mathcal{A}_1} + d_{2,0}(\overline{z})e^{\mathcal{B}_1} = \frac{g_5}{2\partial_z\Phi} + \frac{g_6}{2\overline{\partial_z\Phi}} \quad \text{on}\,\Gamma_0.$$

Finally we set

$$d_{2,\tau} = d_{2,0} + \frac{d_{2,+}e^{2i\tau\psi(\tilde{x})} + d_{2,-}e^{-2i\tau\psi(\tilde{x})}}{2|\det\psi''(\tilde{x})|^{\frac{1}{2}}},$$
$$a_{2,\tau} = a_{2,0} + \frac{a_{2,+}e^{2i\tau\psi(\tilde{x})} + a_{2,-}e^{-2i\tau\psi(\tilde{x})}}{2|\det\psi''(\tilde{x})|^{\frac{1}{2}}}.$$

Construction of the correction term u_{12}. We look for the function u_{12} in the form $u_{12} = u_{-1} + u_0$. The function u_{-1} is given by

$$u_{-1} = \frac{e^{i\tau\psi}}{\tau}\widetilde{\mathcal{R}}_{\tau,B_1}\{e_1g_5\} + \frac{e^{-i\tau\psi}}{\tau}\mathcal{R}_{-\tau,A_1}\{e_1g_6\}$$
$$+\frac{e_2g_5e^{i\tau\psi}}{2\tau^2\partial_z\Phi} + \frac{e_2g_6e^{-i\tau\psi}}{2\tau^2\overline{\partial_z\Phi}}.$$

We set $\varphi = \mathrm{Re}\,\Phi$ and $O_\varepsilon = \{x \in \Omega;\ \mathrm{dist}\,(x,\partial\Omega) \le \varepsilon\}$. For the construction of u_0, first we consider the following boundary value problem

$$L(x,D)w = fe^{\tau\Phi}\ \text{in}\ \Omega, \quad w|_{\Gamma_0} = qe^{\tau\varphi}/\tau \qquad \textbf{[18]}$$

Lemma 1. **A)** *Let $\varepsilon > 0$ be small such that $\overline{O_\varepsilon} \cap (\mathcal{H} \setminus \Gamma_0) = \emptyset$, $f \in L^p(\Omega)$ with $p > 2$ and $q \in H^{\frac{1}{2}}(\Gamma_0)$. There exists a solution of (18) satisfying*

$$\begin{aligned} & |\tau|^{1/2}\|we^{-\tau\varphi}\|_{L^2(\Omega)} + |\tau|^{-1/2}\|(\nabla w)e^{-\tau\varphi}\|_{L^2(\Omega)} \\ + \quad & \|(\nabla w)e^{-\tau\varphi}\|_{L^2(O_\varepsilon)} + |\tau|\|we^{-\tau\varphi}\|_{L^2(O_\varepsilon)} \\ \leq \quad & C_1(\|f\|_{L^p(\Omega)} + \|q\|_{H^{\frac{1}{2}}(\Gamma_0)}) \quad \forall |\tau| \geq \tau_0. \end{aligned}$$

B) *Let $f \in L^2(\Omega)$ and $q = 0$. There exists a solution of (18) satisfying*

$$\begin{aligned} & |\tau|^{1/2}\|we^{-\tau\varphi}\|_{L^2(\Omega)} + |\tau|^{-1/2}\|(\nabla w)e^{-\tau\varphi}\|_{L^2(\Omega)} \\ \leq \quad & C_1\|f\|_{L^2(\Omega)} \quad \forall |\tau| \geq \tau_0. \end{aligned}$$

Here $C_1 > 0$ does not depend on the choices of τ, f, q.

Let $\tilde{w} = a_\tau(z)e^{\mathcal{A}_1+\tau\Phi} + d_\tau(\overline{z})e^{\mathcal{B}_1+\tau\overline{\Phi}} + (u_{11} + u_{-1})e^{\tau\varphi}$. Observe that the function $e^{-\tau\varphi}L_1(x, D)\tilde{w}$ can be represented as a sum of $m_j(\tau, \cdot)$ where

$$\|m_1\|_{L^2(\Omega)} = O(\frac{1}{\tau^2}) \quad \|m_2\|_{L^4(\Omega)} = o(\frac{1}{\tau})$$

and

$$\text{dist}\,(\text{supp}\, m_2, \partial\Omega) > C_3 > 0.$$

Moreover $\|e^{-\tau\varphi}\tilde{w}\|_{H^{\frac{1}{2}}(\Gamma_0)} = o(\frac{1}{\tau^2})$. Then the correction term u_0 can be constructed using Lemma 1.

Carleman estimate. Lemma 1 is derived from the following Carleman estimate with a degenerate weight function.

Lemma 2. *Suppose that Φ satisfies (17). Then there exist τ_0 and C independent of u and τ such that*

$$\begin{aligned} & \|ue^{\tau\varphi}\|^2_{H^1(\Omega)} + \|\frac{\partial u}{\partial\nu}e^{\tau\varphi}\|^2_{L^2(\Gamma_0)} + \tau^2\|\frac{\partial\Phi}{\partial z}ue^{\tau\varphi}\|^2_{L^2(\Omega)} \\ \leq \quad & C_2(\|e^{\tau\varphi}L(x, D)u\|^2_{L^2(\Omega)} + |\tau|\int_{\tilde{\Gamma}}\left|\frac{\partial u}{\partial\nu}\right|^2 e^{2\tau\varphi}d\sigma) \end{aligned} \quad \textbf{[19]}$$

for all $u \in H^1_0(\Omega)$ and all $|\tau| > \tau_0$.

CGO for the adjoint equation. The operator $L_1(x, D)^*$ has the form of the operator $L_1(x, D)$ with different coefficients for the first and zero order terms. Similarly to u_1, we construct the complex geometrical optics solution

$$v(x) = b_\tau(z)e^{\mathcal{B}_2-\tau\Phi} + c_\tau(\overline{z})e^{\mathcal{A}_2-\tau\overline{\Phi}} + v_{11}e^{-\tau\varphi} + v_{12}e^{-\tau\varphi}.$$

Here the functions $\mathcal{A}_2, \mathcal{B}_2 \in C^{6+\alpha}(\overline{\Omega})$ satisfy $2\frac{\partial\mathcal{A}_2}{\partial z} = \overline{A_2}, 2\frac{\partial\mathcal{B}_2}{\partial\overline{z}} = \overline{B_2}$ in Ω, $\text{Im}\,\mathcal{A}_2|_{\Gamma_0} = \text{Im}\,\mathcal{B}_2|_{\Gamma_0} = 0$, and $b_\tau(z) = b(z) + \frac{b_1(z)}{\tau} + \frac{b_{2,\tau}(z)}{\tau^2}, c_\tau(\overline{z}) = c(\overline{z}) + \frac{c_1(\overline{z})}{\tau} + \frac{c_{2,\tau}(\overline{z})}{\tau^2}$. The smooth holomorphic function $b(z)$ and the antiholomorphic function $c(\bar{z})$ have zeros of order five on $\mathcal{H} \setminus \{\tilde{x}\}$, are not equal to zero at $\tilde{x}$ and satisfy the boundary condition $b(z)e^{\mathcal{B}_2} + c(\overline{z})e^{\mathcal{A}_2} = 0$ on Γ_0.

Using the phase function Φ constructed in Proposition 9 we compute the right hand side of (15) up to the terms of order $\frac{1}{\sqrt{\tau}}$.

$$\begin{aligned} 0 = O(\frac{1}{\tau}) + \tau F_1 + F_0 + \\ \sum_{x\in\mathcal{G}\setminus x_\pm}((\frac{2\pi}{i\frac{\partial^2\psi}{\partial\overline{\tau}^2}(x)})^{\frac{1}{2}}(\overline{c}d(B_1 - B_2))(x)\frac{e^{(\mathcal{B}_1+\overline{\mathcal{A}_2}-2\tau i\psi)(x)}}{\sqrt{\tau}} \\ +(\frac{2\pi}{-i\frac{\partial^2\psi}{\partial\overline{\tau}^2}(x)})^{\frac{1}{2}}(a\overline{b}(A_1 - A_2))(x)\frac{e^{(\mathcal{A}_1+\overline{\mathcal{B}_2}+2\tau i\psi)(x)}}{\sqrt{\tau}}). \end{aligned}$$

Here F_0 and F_1 are independent of τ. This immediately implies (7). Moreover the equation $F_1 = 0$ implies that there exist a holomorphic function $\Theta \in H^{\frac{1}{2}}(\Omega)$ and an antiholomorphic function $\tilde{\Theta} \in H^{\frac{1}{2}}(\Omega)$ such that

$$\Theta|_{\tilde{\Gamma}} = e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}, \quad \tilde{\Theta}|_{\tilde{\Gamma}} = e^{\mathcal{B}_1+\overline{\mathcal{B}_2}} \quad \textbf{[20]}$$

and

$$e^{\mathcal{B}_1+\overline{\mathcal{B}_2}}\Theta - e^{\mathcal{A}_1+\overline{\mathcal{A}_2}}\tilde{\Theta} = 0 \quad \text{on } \Gamma_0. \quad \textbf{[21]}$$

Computing the asymptotic formula of the right hand side of (15) with an error up to the order $o(\frac{1}{\tau})$ and using (20), (21) we have

$$\begin{aligned} o(\frac{1}{\tau}) = \sum_{k=1}^{3}\tau^{2-k}\widetilde{F}_k + \quad \textbf{[22]} \\ -\frac{\pi}{\tau}\left\{\mathcal{Q}_+a\overline{b}e^{(\mathcal{A}_1+\overline{\mathcal{B}_2}+2\tau i\psi)} + \mathcal{Q}_-d\overline{c}e^{(\mathcal{B}_1+\overline{\mathcal{A}_2}-2i\tau\psi)}\right\}(\tilde{x}) \\ - \quad \frac{2\pi e^{-2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}}\frac{\overline{\partial g_4(\tilde{x})}}{\partial z}e^{-\overline{\mathcal{B}_2}(\tilde{x})}(d\tilde{\Theta})(\tilde{x}) \\ + \quad \frac{2\pi e^{-2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}}\frac{\partial g_1(\tilde{x})}{\partial z}e^{-\mathcal{A}_1(\tilde{x})}(\overline{c}\Theta)(\tilde{x}) \\ - \quad \frac{2\pi e^{2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}}\frac{\overline{\partial g_3(\tilde{x})}}{\partial\overline{z}}e^{-\overline{\mathcal{A}_2}(\tilde{x})}(a\Theta)(\tilde{x}) \\ + \quad \frac{2\pi e^{2i\tau\psi(\tilde{x})}}{\tau|\det\psi''(\tilde{x})|^{\frac{1}{2}}}\frac{\partial g_2(\tilde{x})}{\partial\overline{z}}e^{-\mathcal{B}_1(\tilde{x})}(\overline{b}\tilde{\Theta})(\tilde{x}), \end{aligned}$$

where $\mathcal{Q}_+ = -(B_1 - B_2)A_1 - (A_1 - A_2)B_2 - 2\frac{\partial}{\partial z}(A_1 - A_2) + (q_1 - q_2)$, $\mathcal{Q}_- = -(A_1 - A_2)B_1 - (B_1 - B_2)A_2 - 2\frac{\partial}{\partial\overline{z}}(B_1 - B_2) + (q_1 - q_2)$ and $\widetilde{F}_k$ are some constants independent of τ.

Let η be a smooth function such that η is zero in some neighborhood of $\partial\Omega$ and $\eta(\tilde{x}) \neq 0$. Observe that the partial Cauchy data of the operator $L_2(x, D)$ and the operator $e^{-s\eta}L_1(x, D)e^{s\eta}$ are exactly the same. Therefore we have the analog of (22) for these two operators with $\mathcal{A}_1$ and $\mathcal{B}_1$ replaced by $\mathcal{A}_1 - s\eta$ and $\mathcal{B}_1 - s\eta$. The coefficients A_1, B_1 should be replaced by $A_1 + 2s\frac{\partial\eta}{\partial\overline{z}}, B_1 + 2s\frac{\partial\eta}{\partial z}$. The functions $\mathcal{Q}_\pm$ will not change. The function q_1 should be replaced by $q_1 + s\Delta\eta + s^2|\nabla\eta|^2 + 2sA_1\frac{\partial\eta}{\partial z} + 2sB_1\frac{\partial\eta}{\partial\overline{z}}$. This immediately implies that $(\mathcal{Q}_+a\overline{b})(\tilde{x}) = (\mathcal{Q}_-d\overline{c})(\tilde{x}) = 0$. By Proposition 8 we construct the set of functions Φ_ϵ satisfying (17) such that the union of the sets of the critical points of these functions is dense in Ω. This finishes the proof of (8) and (9). The proof of the theorem is completed. $\square$

4 Sketch of Proof of Theorem 1

For simplicity we restrict ourselves to the case that Ω is simply connected. Suppose that the two operators

$$L_j(x, D) = \Delta_{g_j} + 2A_j\frac{\partial}{\partial z} + 2B_j\frac{\partial}{\partial\overline{z}} + q_j$$

generate the same partial Cauchy data. Multiplying the metric g_2, if necessary, by some positive smooth function $\tilde{\beta}$, we may assume that

$$\frac{\partial^\ell}{\partial\nu^\ell}(g_1^{jk} - g_2^{jk})|_{\tilde{\Gamma}} = 0, \ell \in \{0, 1\}. \quad \textbf{[23]}$$

Observe that without loss of generality, we may assume that there exists a smooth positive function μ_2 such that $g_2 =$

$\mu_2 I$. Indeed, using isothermal coordinates we make a change of variables in the operator $L_2(x,D)$ such that $g_2=\mu_2 I$. Then we make the same changes of variables in the operator $L_1(x,D)$. The partial Cauchy data for both operators obtained by this change of variables are the same.

Let ω be a subdomain in $\mathbb{R}^2$ such that $\Omega\cap\omega=\emptyset$, $\partial\omega\cap\partial\Omega=\tilde\Gamma$ and the boundary of the domain $\tilde\Omega=\mathrm{Int}(\Omega\cup\omega)$ is smooth. We extend μ_2 in $\tilde\Omega$ as a smooth positive function and set $g_1^{-1}=\frac{1}{\mu_2}I$ in ω. By (23) $g_1\in C^1(\bar\Omega)$.

There exists an isothermal mapping $\chi_1=(\chi_{1,1},\chi_{1,2})$ such that the operator $L_1(x,D)$ is transformed to the following form:

$$Q_1(y,D)=\frac{1}{\mu_1}\Delta+2C_1\frac{\partial}{\partial z}+2D_1\frac{\partial}{\partial\bar z}+r_1 \quad y\in\chi_1(\tilde\Omega), \qquad \mathbf{[24]}$$

where μ_1 is a smooth positive function in $\chi_1(\tilde\Omega)$ and C_1,D_1,r_1 are some smooth complex valued functions. Consider a solution to the problem

$$Q_1(y,D)w=0 \quad \text{in } \chi_1(\tilde\Omega),\ \ w|_{\chi_1(\Gamma_0)}=0$$

of the form (16) with the holomorphic weight function Φ_1. Then the function $u_1(x)=w(\chi_1(x))$ satisfies

$$L_1(x,D)u_1=0 \quad \text{in } \tilde\Omega, \quad u_1|_{\Gamma_0}=0.$$

Since the partial Cauchy data for the operators $L_1(x,D)$ and $L_2(x,D)$ are the same, there exists a function u_2 such that

$$L_2(x,D)u_2=0 \quad \text{in } \Omega, \quad u_2|_{\Gamma_0}=0,\ (\frac{\partial u_1}{\partial\nu_{g_1}}-\frac{\partial u_2}{\partial\nu_{g_2}})|_{\tilde\Gamma}=0. \qquad \mathbf{[25]}$$

Using (23) and (25), we extend u_2 on $\tilde\Omega$ such that

$$u_1|_\omega=u_2|_\omega. \qquad \mathbf{[26]}$$

Let φ_2 be the harmonic function in $\tilde\Omega$ such that

$$\frac{\partial\varphi_2}{\partial\nu}|_{\Gamma_0}=0, \quad \varphi_2=\mathrm{Re}\,\Phi_1\circ\chi_1 \quad \text{in } \partial\tilde\Omega\setminus\Gamma_0.$$

We claim that

$$\varphi_2=\mathrm{Re}\,\Phi_1\circ\chi_1 \quad \text{in } \omega. \qquad \mathbf{[27]}$$

Thanks to the Carleman estimate (19) there exists $\tau_0=\tau_0(\epsilon)$ such that

$$\|e^{-\tau\varphi_2}u_2\|_{L^2(\tilde\Omega_\epsilon)}\le C_0|\tau e^{\delta_\epsilon|\tau|}| \quad \forall|\tau|\ge\tau_0, \qquad \mathbf{[28]}$$

where $C_0=C_0(\epsilon)$ is independent of τ and $\delta_\epsilon\to 0$ as $\epsilon\to 0$. On the other hand $u_1=e^{\tau\mathrm{Re}\,\Phi_1\circ\chi_1}(a_\tau e^{\mathcal{C}_1+i\tau\mathrm{Im}\,\Phi_1}+b_\tau e^{\mathcal{D}_1-i\tau\mathrm{Im}\,\Phi_1})\circ\chi_1+O(\frac{1}{\tau}))$. Here $2\frac{\partial\mathcal{C}_1}{\partial\bar z}=-C_1$, $2\frac{\partial\mathcal{D}_1}{\partial z}=-D_1$, $\mathrm{Im}\,\mathcal{C}_1|_{\Gamma_0}=\mathrm{Im}\,\mathcal{D}_1|_{\Gamma_0}=0$. Then by (26) the following holds true:

$$e^{\tau\varphi_2}(e^{-\tau\varphi_2}u_2)=e^{\tau\mathrm{Re}\,\Phi_1\circ\chi_1}((a_\tau e^{\mathcal{C}_1+i\tau\mathrm{Im}\,\Phi_1} \qquad \mathbf{[29]}$$

$$+b_\tau e^{\mathcal{D}_1-i\tau\mathrm{Im}\,\Phi_1})\circ\chi_1+O(\frac{1}{\tau})) \quad \forall x\in\omega.$$

This equality implies (27) immediately. Indeed, let for some point $\hat x$ from ω

$$\varphi_2(\hat x)\ne\mathrm{Re}\,\Phi_1\circ\chi_1(\hat x). \qquad \mathbf{[30]}$$

Then there exists a ball $B(\hat x,\delta')\equiv\{x\in\mathbf{R}^2;\ |x-\hat x|<\delta'\}\subset\omega$ such that

$$|\varphi_2(x)-\mathrm{Re}\,\Phi_1\circ\chi_1(x)|>\alpha'>0 \quad \forall x\in\overline{B(\hat x,\delta')}. \qquad \mathbf{[31]}$$

Let us fix positive ϵ_1 such that $\overline{B(\hat x,\delta')}\subset\Omega_{\epsilon_1}$ and $2\delta_{\epsilon_1}<\alpha'$. Form (29) by (28) and (31) we have

$$\begin{aligned} & C'e^{|\tau|\alpha'}Vol(B(\hat x,\delta'))^{\frac12}\\ \le\ & \|e^{\tau(\mathrm{Re}\,\Phi_1\circ\chi_1-\varphi_2)}(((a_\tau e^{\mathcal{C}_1+i\tau\mathrm{Im}\,\Phi_1}+b_\tau e^{\mathcal{D}-i\tau\mathrm{Im}\,\Phi_1})\circ\chi_1\\ +\ & O(\frac{1}{\tau}))\|_{L^2(B(\hat x,\delta'))}\\ =\ & \|e^{-\tau\varphi_2}u_2\|_{L^2(B(\hat x,\delta'))}\le C_0|\tau|e^{\delta_\epsilon|\tau|}, \end{aligned}$$

where $\tau>\tau_0$ if $\varphi_2(\hat x)<\mathrm{Re}\,\Phi_1\circ\chi_1(\hat x)$ and $\tau<-\tau_0$ if $\varphi_2(\hat x)>\mathrm{Re}\,\Phi_1\circ\chi_1(\hat x)$. The above inequality contradicts (30).

Let $\Xi=\chi_{1,1}+i\chi_{1,2}$. Using the Cauchy-Riemann equations we construct a harmonic function ψ_2 such that the function $\Phi_2=\varphi_2+i\psi_2$ is holomorphic in $\tilde\Omega$. Moreover we take the function Φ_1 which may be holomorphically extended to some domain $\mathcal{O}$ such that $\chi_1(\tilde\Omega)\subset\mathcal{O}$. Observe that $\Phi_2=\Phi_1\circ\Xi$ in ω. Then $\Xi=\Phi_1^{-1}\circ\Phi_2$ in ω. The function Ξ may be extended up to a single valued holomorphic function $\tilde\Xi$ in $\tilde\Omega$ such that $\tilde\Xi:\tilde\Omega\to\chi_1(\tilde\Omega)$ and $\tilde\Xi(\tilde\Omega)=\chi_1(\tilde\Omega)$.

In Ω, consider the new infinitesimal coordinates for the operator L_1 given by the mapping $\tilde\Xi^{-1}\circ\Xi(x)$. In these coordinates, the operator $L_1(x,D)$ has the form

$$\tilde Q(x,D)=\frac{1}{\tilde\mu_1}\Delta+2\tilde A_1\frac{\partial}{\partial z}+2\tilde B_1\frac{\partial}{\partial\bar z}+\tilde q_1. \qquad \mathbf{[32]}$$

Since $\tilde\Xi^{-1}\circ\Xi(x)|_{\tilde\Gamma}=Id$, the partial Cauchy data for the operators $L_2(x,D)$ and $\tilde Q(x,D)$ are exactly the same. The operators $L_2(x,D)$ and $\tilde Q(x,D)$ are particular cases of the operator (1). Since $(\mu_2-\tilde\mu_1)|_{\tilde\Gamma}=0$, the Cauchy data $\mathcal{C}_{\mu_2 I,A_2,B_2,q_2}$ and $\mathcal{C}_{\tilde\mu_1 I,\tilde A_1,\tilde B_1,\tilde q_1}$ are equal. We multiply the operator $\tilde Q(x,D)$ by the function $\tilde\mu_1/\mu_2$ and denote the resulting operator as $\hat Q(x,D)=\frac{\tilde\mu_1}{\mu_2}\tilde Q(x,D)$. Therefore by Corollary 4 there exists a function η which satisfies (3) such that $L_2(x,D)=e^{-\eta}\hat Q(x,D)e^{\eta}$. The proof of the theorem is completed. $\square$

1. Ahlfors L (1966) Quasiconformal mappings Princeton, NJ
2. Astala K, Päivärinta L (2006) Calderón's inverse conductivity problem in the plane. Ann of Math 163: 265–299.
3. Astala K, Lassas M, Päiväirinta L (2005) Calderón's inverse problem for anisotropic conductivity in the plane. Comm Partial Differential Equations 30: 207–224.
4. Bukhgeim A (2008) Recovering the potential from Cauchy data in two dimensions. J Inverse Ill-Posed Probl 16: 19–34.
5. Calderón A P (1980) On an inverse boundary value problem. in Seminar on Numerical Analysis and its Applications to Continuum Physics Soc Brasil Mat Río de Janeiro: 65–73.
6. Cheng J, Yamamoto M (2004) Determination of two convection coefficients from Dirichlet to Neumann map in the two-dimensional case. SIAM J Math Anal 35: 1371–1393.
7. Guillarmou C, Tzou L Calderón inverse problem with partial data on Riemann surfaces preprint arXiv:0908.1417.
8. Imanuvilov O, Uhlmann G. Yamamoto M (2010) The Calderón problem with partial data in two dimensions. J. Amer. Math. Soc. 23: 655-691.
9. Kang H, Uhlmann G (2004) Inverse problems for the Pauli Hamiltonian in two dimensions. Journal of Fourier Analysis and Applications 10: 201-215.
10. Kohn R, Vogelius M (1984) Identification of an unknown conductivity by means of measurements at the boundary, in Inverse Problems, edited by D. McLaughlin, SIAM-AMS Proceedings, 14 113-123.
11. Nachman A (1996) Global uniqueness for a two-dimensional inverse boundary value problem. Ann of Math 143: 71–96.
12. Sun Z (1993) An inverse boundary value problem for the Schrödinger operator with vector potentials in two dimensions. Comm Partial Differential Equations 18: 83–124.

13. Sun Z, Uhlmann G (2003) Anisotropic inverse problems in two dimensions. Inverse Problems 19: 1001-1010.
14. Temam R (2001) Navier-Stokes equations, Theory and numerical analysis, AMS Chelsia, Providense.