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UNIQUE CONTINUATION PROPERTY FOR THE ELASTICITY WITH GENERAL RESIDUAL STRESS

Dedicated to David Colton and Rainer Kress on the occasion of their 65th birthday

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Abstract. We prove the unique continuation property for the isotropic elasticity system with *arbitrarily large* residual stress. This work improves the result obtained in [10] where the residual stress is assumed to be small.

1. **Introduction.**

In this paper we prove the unique continuation property (UCP) for the isotropic elasticity system with residual stress. Residual stresses are stresses that remain after the original cause of the stresses has been removed. We consider the case of large residual stresses. We formulate the mathematical problem and the result more precisely below.

Let Ω be a connected open domain in $\mathbb{R}^n$, we consider the following time-harmonic elasticity system

$$\nabla \cdot \sigma + \rho\omega u = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\sigma = (\sigma_{ij})_{i,j=1}^n$ is the stress tensor field, $\rho(x)$ is the density function, and $\omega \in \mathbb{C}$ is the frequency. The vector-valued function $u(x) = (u_i(x))_{i=1}^n$ is the displacement vector. Here we assume that the stress tensor σ is given by

$$\sigma(x) = T(x) + (\nabla u)T(x) + \lambda(x)(\mathrm{tr}E)I + 2\mu(x)E, \tag{1.2}$$

where $E(x) = (\nabla u + \nabla u^t)/2$ is the infinitesimal strain and $\lambda(x), \mu(x)$ are the Lamé parameters. The tensor $T(x) = (t_{ij}(x))_{i,j=1}^n$ represents the residual stress, which satisfies

$$t_{ij}(x) = t_{ji}(x) \quad \forall\ 1 \le i, j \le n \text{ and } x \in \Omega$$

and

$$\nabla \cdot T = 0 \quad \forall\ x \in \Omega.$$

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The expression (1.2) is a simple constitutive equation modelling the linear elasticity with residual stress, which was considered in several inverse problems (see [4], [12] and [13] for example). A more general constitutive equation in linear elasticity with residual stress is given as

$$\sigma = T + (\nabla u)T + L(E),$$

where $L(E)$ is the incremental elasticity tensor. Some explicit forms of $L(E)$ were derived in [5] and [8].

Here we are concerned with the UCP for (1.1), namely, if $u \in H^2_{loc}(\Omega)$ is a solution (1.1) and $u(x) = 0$ in a non-empty open subset of Ω, the $u(x) \equiv 0$ in Ω. The UCP implies the Runge approximation property that has been widely used in inverse problems starting with the work of [7] and [6]. It has been also used for detection of elastic cracks and inclusions, see [9] for a review.

Under the assumption that the residual stress is sufficiently small, the UCP for (1.1) has been proved in [10]. The purpose of this paper is to remove this smallness assumption. We will prove the UCP for (1.1) assuming the usual *strong ellipticity* condition which we proceed to define. If we define the elastic tensor $C = (C_{ijkl})_{i,j,k,l=1}^n$ with

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + t_{jl}\delta_{ik},$$

then (1.1) is equivalent to

$$\nabla \cdot (C\nabla u) + \rho\omega u = 0.$$

The strong ellipticity condition is described as follows: there exists a constant $\gamma > 0$ such that for all vectors $a = (a_j)_{j=1}^n$ and $b = (b_j)_{j=1}^n$

$$\sum_{ijkl} C_{ijkl} a_i b_j a_k b_l \ge \gamma |a|^2 |b|^2 \quad \forall\ x \in \Omega,$$

which is equivalent to that

$$A_1(x, D) := \sum_{jk} (\mu\delta_{jk} + t_{jk})\partial^2_{x_j x_k} \quad \text{and} \quad A_2(x, D) := \sum_{jk} ((\lambda + 2\mu)\delta_{jk} + t_{jk})\partial^2_{x_j x_k}$$

are uniform elliptic operators. In other words, we assume that there exists $\theta > 0$ such that for any vector $\xi = (\xi_j)_{j=1}^n$

$$\sum_{jk} t_{jk}\xi_j\xi_k + \mu|\xi|^2 \ge \theta|\xi|^2 \quad \text{and} \quad \sum_{jk} t_{jk}\xi_j\xi_k + (\lambda + 2\mu)|\xi|^2 \ge \theta|\xi|^2 \tag{1.3}$$

for all $x \in \Omega$. The standard linear elasticity with Lamé coefficients λ, μ is uniformly elliptic if

$$\mu \ge \theta \quad \text{and} \quad \lambda + 2\mu \ge \theta. \tag{1.4}$$

In view of (1.4), (1.3) holds for any semi-positive definite residual stress.

We will follow the lines in [10] to establish the UCP. The main difficulty is to generalize two Carleman estimates derived in [10] to the case of large residual stress. The details are carried out in Section 3. We refer the reader to [9] and references therein for related literature on the UCP in elasticity.

2. **Proof of the UCP.** In this section we will prove the UCP based on Carleman estimates. In addition to (1.3), we assume that

$$\lambda(x),\ \mu(x),\ t_{jk}(x)\ (1 \le j,k \le n) \in W^{2,\infty}(\Omega),\ \text{and}\ \rho(x) \in W^{1,\infty}(\Omega).$$

Then we can rewrite (1.1) in the form

$$A_1 u + (\lambda+\mu)\nabla(\nabla\cdot u) = P_1(x,D)u \quad \text{for } x\in\Omega, \tag{2.1}$$

where $P_1(x,D)$ is a first order differential operator with $W^{1,\infty}(\Omega)$ coefficients. Now we define a scalar function $v(x) := \nabla\cdot u$ and derive from (2.1) that

$$A_1 u = -(\lambda+\mu)\nabla v + P_1(x,D)u \quad \text{for } x\in\Omega.$$

Applying the divergence on both sides of (2.1) yields

$$A_2 v = -2\sum_i (\partial_{x_i}\mu)\Delta u_i - \sum_{ijk}(\partial_{x_i} t_{jk})\partial^2_{x_j x_k} u_i + Q_1(x,D)(u,v) \quad \text{in } \Omega,$$

where $Q_1(x,D)$ is a first order differential operator acting on u and v with $L^\infty(\Omega)$ coefficients. Therefore, to prove the UCP for (1.1), it suffices to prove the UCP for

$$\begin{cases} A_1 u = \widetilde{P}_1(x,D)(u,v) \\ A_2 v = Q_2(x,D)u + Q_1(x,D)(u,v), \end{cases} \tag{2.2}$$

where $Q_2(x,D)$ is a second order differential operator acting on u with $W^{1,\infty}(\Omega)$ coefficients and $\widetilde{P}_1(x,D)$ is a first order differential operator acting u,v with $W^{1,\infty}(\Omega)$ coefficients. Note that the system (2.2) does not have a decoupled principal part. It should be pointed out that one can eliminate $Q_2 u$ in (2.2), when there is no residual stress (see [1] or [2]).

The proof of the UCP for (2.2) relies on two Carleman estimates. We state the estimates here and will derive them in the following section. For simplicity, we set $A = \sum_{jk} a_{jk}(x)\partial^2_{x_j x_k}$ with $a_{jk}(x) = a_{kj}(x)$, $a_{ij} \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ for $1 \le i,j \le n$, and for any fixed compact set $K \subset \mathbb{R}^n$

$$\sum_{jk} a_{jk}(x)\xi_j\xi_k \ge \theta|\xi|^2 \quad \forall\ \xi\in\mathbb{R}^n,\ x\in K. \tag{2.3}$$

Let $r_0 < 1$ and $U_{r_0} = \{u \in C_0^\infty(\mathbb{R}^n\setminus\{0\}) : \operatorname{supp}(u) \subset B_{r_0}\}$, where B_{r_0} is the ball centered at the origin with radius r_0. Denote $r = |x|$, $\psi = \exp(r^{-\beta})$ and $s = s_0 + \tilde{c}\beta$ with $s_0, \tilde{c} \in \mathbb{R}$.

Proposition 2.1. *There exists a positive constant β_0 such that for all $\beta \ge \beta_0$ and $u \in U_{r_0}$ with r_0 sufficiently small, we have that*

$$\int r^{-s}\psi^2 \sum_{jk} |\partial^2_{x_j x_k} u|^2 dx \le c\int r^{-s}\psi^2(\beta^2 r^{-2\beta-2}|\nabla u|^2 + |Au|^2)dx, \tag{2.4}$$

where the constant c is independent of β and u.

Proposition 2.2. *There exists a positive constant β_0 such that for all $\beta \ge \beta_0$ and $u \in U_{r_0}$ with r_0 small enough, we have*

$$\beta^2\int r^{-s-\beta-1}\psi^2(|\nabla u|^2 + |u|^2)dx \le c\int r^{-s}\psi^2|Au|^2 dx. \tag{2.5}$$

The constant c is independent of β and u

Assuming the estimates (2.4) and (2.5), we can now prove the UCP for (1.1).

Theorem 2.3. *Assume* $\lambda(x), \mu(x), t_{jk}$ $(1 \leq j, k \leq n) \in W^{2,\infty}(\Omega)$ *and* $\rho(x) \in W^{1,\infty}(\Omega)$. *Let the strong ellipticity condition* (1.3) *hold, then for any* $H^2_{loc}(\Omega)$ *solution* u *of* (1.1) *satisfying* $u = 0$ *in a non-empty open subset of* Ω, *we have* $u \equiv 0$ *in* Ω.

Proof. We will prove this theorem using similar arguments to [10] and [14]. To make this paper self-contained, we include the proof here. Let (u, v) vanish in a neighborhood of $x_0 \in \Omega$. Without loss of generality, we assume $x_0 = 0$. We set $\tilde{r} = \min\{r_0, 1/2, \mathrm{dist}(0, \partial\Omega)\}$. Now let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function satisfying $0 \leq \chi \leq 1$, $\chi|_{B_{\tilde{r}/2}} = 1$ and $\mathrm{supp}\,(\chi) \subset B_{\tilde{r}}$. Denote $w_1 = \chi u$ and $w_2 = \chi v$. From (2.2) we have that

$$\begin{array}{l} |A_1 w_1| \leq c(e(w_1) + e(w_2))^{1/2} + f_1, \\ |A_2 w_2| \leq c[\sum\limits_{ij} |\partial^2_{x_i x_j} w_1| + (e(w_1) + e(w_2))^{1/2}] + f_2, \end{array} \tag{2.6}$$

where $e(w) = |\nabla w|^2 + |w|^2$ and f_j is supported in $B_{\tilde{r}} \setminus B_{\tilde{r}/2}$ for $j = 1, 2$. It follows from (2.6) that

$$\begin{array}{l} I := \gamma \int r^{-\beta} \psi^2 |A_1 w_1|^2 dx + \int r \psi^2 |A_2 w_2|^2 dx \\ \leq c(F + G + \int r \psi^2 \sum\limits_{ij} |\partial^2_{x_i x_j} w_1|^2 dx), \end{array} \tag{2.7}$$

where

$$\begin{array}{l} F = \gamma \int r^{-\beta} \psi^2 f_1^2 dx + \int r \psi^2 f_2^2 dx, \\ G = \int (r + \gamma r^{-\beta}) \psi^2 (e(w_1) + e(w_2)) dx. \end{array}$$

and γ is a large parameter which will be chosen later. Now we want to apply (2.4) and (2.5) to $A_1 w_1$ and $A_2 w_2$. Taking $s = -1$ (i.e., $s_0 = -1, \tilde{c} = 0$) in the estimate (2.4) for $A_1 w_1$ and substituting it into (2.7) yield

$$I \leq c(F + G + \int r \psi^2 |A_1 w_1|^2 dx + \beta^2 \int r^{-2\beta-1} \psi^2 |\nabla w_1|^2 dx). \tag{2.8}$$

Estimating the last term of (2.8) using (2.5) for $A_1 w_1$ with $s = \beta$, we obtain that

$$I \leq c(F + G + \int r^{-\beta} \psi^2 |A_1 w_1|^2 dx). \tag{2.9}$$

Now taking γ sufficiently large, we can absorb the last term of (2.9) by the same term in I and get

$$I \leq c(F + G). \tag{2.10}$$

From now on we fix the parameter γ.

Next using $s = \beta$ in (2.5) for $A_1 w_1$ and $s = -1$ in (2.5) for $A_2 w_2$ we obtain

$$\begin{array}{rl} H: & = \beta^2 \int r^{-2\beta-1} \psi^2 e(w_1) dx + \beta^2 \int r^{-\beta} \psi^2 e(w_2) dx \\ & \leq c(\int r^{-\beta} \psi^2 |A_1 w_1|^2 dx + \int r \psi^2 |A_2 w_2|^2 dx). \end{array} \tag{2.11}$$

Combining (2.10) and (2.11) gives

$$H \leq c(F + G) \leq c(F + \int (r + \gamma r^{-\beta}) \psi^2 (e(w_1) + e(w_2)) dx). \tag{2.12}$$

Now observing that $r < r^{-\beta} < \beta r^{-\beta} < \beta r^{-2\beta-1}$ when $r \leq \tilde{r}$ and $\beta > 1$, we obtain from (2.12) that

$$H \leq c(F + \beta \int r^{-2\beta-1} \psi^2 e(w_1) dx + \beta \int r^{-\beta} \psi^2 e(w_2) dx). \tag{2.13}$$

Taking β sufficiently large in (2.13), we get that

$$H \le cF,$$

i.e.

$$\beta^2 \int r^{-2\beta-1}\psi^2 e(w_1)dx + \beta^2 \int r^{-\beta}\psi^2 e(w_2)dx \le c(\int r^{-\beta}\psi^2 f_1^2 dx + \int r\psi^2 f_2^2 dx)$$

from which we immediately have

$$\beta^2 \int_{B_{\tilde{r}/2}} r^{-\beta}\psi^2(w_1^2+w_2^2)dx \le c\int_{B_{\tilde{r}}\setminus B_{\tilde{r}/2}} r^{-\beta}\psi^2(f_1^2+f_2^2)dx. \tag{2.14}$$

Since $r^{-\beta}\psi^2$ is a strictly decreasing function, (2.14) implies that

$$\beta^2 \int_{B_{\tilde{r}/2}} (w_1^2+w_2^2)dx \le c\int_{B_{\tilde{r}}\setminus B_{\tilde{r}/2}} (f_1^2+f_2^2)dx$$

and therefore $(w_1, w_2) = 0$ on $B_{\tilde{r}/2}$ if we choose β sufficiently large. Clearly, (u, v) must be zero throughout Ω. □

3. **The Carleman estimates.** This section is devoted to the proof of (2.4) and (2.5). The use of weight function $\exp(r^{-\beta})$ in Carleman estimates dated back to Protter [11], see also [3, Chapter 8.3].

Proof of Proposition 2.1. Let $A_0u = \sum_{jk} a_{jk}(0)\partial^2_{x_jx_k}u$. In view of (2.3), the symmetric matrix $(a_{jk}(0))_{jk=1}^n =: B$ is positive-definite. Therefore, there exists an orthogonal matrix M such that

$$MBM^t = \mathrm{diag}(\alpha_1,\cdots,\alpha_n),$$

where $\alpha_n \ge \cdots \ge \alpha_1 > 0$. In the new coordinate system $y = Mx$, A_0u becomes

$$\widetilde{A}_0 w := \sum_j \alpha_j \partial^2_{y_jy_j} w,$$

where $w(y) = u(M^{-1}y)$. It is clear that $u \in U_{r_0}$ if and only if $w \in U_{r_0}$. Now we would like to prove (2.4) for $\widetilde{A}_0$. Note that $r = |x| = |M^{-1}y| = |y|$ and $\psi(x) = \psi(y)$. Let $w \in U_{r_0}$, we compute

$$\begin{aligned}
&\textstyle\int r^{-s}\psi^2|\widetilde{A}_0w|^2dy\\
&= \textstyle\int r^{-s}\psi^2\sum_{jk}\alpha_j\alpha_k\partial^2_{y_jy_j}w\partial^2_{y_ky_k}wdy\\
&= -\textstyle\int\sum_{jk}\alpha_j\alpha_k\partial_{y_k}(r^{-s}\psi^2)\partial^2_{y_jy_j}w\partial_{y_k}wdy - \int\sum_{jk}\alpha_j\alpha_k r^{-s}\psi^2\partial^3_{y_jy_jy_k}w\partial_{y_k}wdy\\
&= \textstyle\int\sum_{jk}\alpha_j\alpha_k(sr^{-s-2}+2\beta r^{-s-\beta-2})y_k\psi^2\partial^2_{y_jy_j}w\partial_{y_k}wdy\\
&\quad -\textstyle\int\sum_{jk}\alpha_j\alpha_k(sr^{-s-2}+2\beta r^{-s-\beta-2})y_j\psi^2\partial^2_{y_jy_k}w\partial_{y_k}wdy\\
&\quad +\textstyle\int\sum_{jk}\alpha_j\alpha_k r^{-s}\psi^2|\partial^2_{y_jy_k}w|^2dy\\
&\ge -\varepsilon\textstyle\int r^{-s}\psi^2\sum_{jk}|\partial^2_{y_jy_k}w|^2dy - c(\varepsilon)\int(s^2r^{-s-2}+\beta^2r^{-s-2\beta-2})\psi^2|\nabla w|^2dy\\
&\quad +\alpha_1^2\textstyle\int\sum_{jk}r^{-s}\psi^2|\partial^2_{y_jy_k}w|^2dy,
\end{aligned} \tag{3.1}$$

where we have used the inequality

$$|ab| \le \varepsilon a^2 + c(\varepsilon)b^2 \quad \text{for } \varepsilon > 0.$$

Taking ε sufficiently small ($\alpha_1^2 - \varepsilon > 0$) and β large enough, we get from (3.1) that

$$\int r^{-s}\psi^2\sum_{jk}|\partial^2_{y_jy_k}w|^2dy \le c\int r^{-s}\psi^2(\beta^2r^{-2\beta-2}|\nabla w|^2 + |\widetilde{A}_0w|^2)dy. \tag{3.2}$$

Now returning to the original coordinates x, we find that

$$\widetilde{A}_0 w = A_0 u$$

and

$$\partial^2_{x_i x_j} u = \sum_{kl} m_{ki} m_{lj} \partial^2_{y_k y_l} w \quad 1 \le i, j \le n,$$

where $M = (m_{ij})_{ij=1}^n$. Note that $(\sum_{ij} |\partial^2_{x_i x_j} u|^2)^{1/2}$ is the Frobenius norm of the matrix $(\partial^2_{x_i x_j} u)_{ij=1}^n$ and it is preserved by orthogonal transformations. Therefore, (3.2) implies that

$$\int r^{-s}\psi^2 \sum_{jk} |\partial^2_{x_j x_k} u|^2 dx \le c \int r^{-s}\psi^2 (\beta^2 r^{-2\beta-2} |\nabla u|^2 + |A_0 u|^2) dx. \tag{3.3}$$

For the variable coefficients case, since $a_{jk}(x) \in W^{1,\infty}_{loc}(\mathbb{R}^n)$, $a_{jk}(x)$ is Lipschitz in B_{r_0} for any $r_0 > 0$, $1 \le j, k \le n$. In other words, we have that

$$|a_{jk}(x) - a_{jk}(0)| \le c_0 |x| \quad \forall\, x \in B_{r_0}, \tag{3.4}$$

where $c_0 > 0$ is a constant. Combining (3.3) and (3.4), we obtain that

$$\begin{array}{l} \int r^{-s}\psi^2 \sum_{jk} |\partial^2_{x_j x_k} u|^2 dx \\ \le c \int r^{-s}\psi^2 (\beta^2 r^{-2\beta-2} |\nabla u|^2 + |A_0 u|^2) dx \\ \le c \int r^{-s}\psi^2 (\beta^2 r^{-2\beta-2} |\nabla u|^2 + |(A - A_0) u|^2 + |Au|^2) dx \\ \le c \int r^{-s}\psi^2 (\beta^2 r^{-2\beta-2} |\nabla u|^2 + r_0^2 \sum_{jk} |\partial^2_{x_j x_k} u|^2 + |Au|^2) dx. \end{array} \tag{3.5}$$

So by choosing r_0 small enough in (3.5), we get (2.4). □

Proof of Proposition 2.2. For simplicity, we denote $\partial_{x_j} = \partial_j$. Let $\phi = \psi^{-1}$ and $u = r^{\tau/2}\phi z$, then

$$\begin{array}{rl} r^{-s/2}\psi A u & = r^{-s/2}\psi A(r^{\tau/2}\phi z) \\ & = r^{-s/2}\psi [r^{\tau/2}\phi A z + 2\sum_{ij} a_{ij} \partial_i z \partial_j (r^{\tau/2}\phi) + z A(r^{\tau/2}\phi)]. \end{array}$$

By virtue of the inequality $(a + b + c)^2 \ge 2ab + 2bc$, we have that

$$\begin{array}{rl} \int r^{-s}\psi^2 |Au|^2 dx & \ge 4 \int r^{-s}\psi^2 \sum_{ij} a_{ij} \partial_i z \partial_j (r^{\tau/2}\phi) r^{\tau/2}\phi A z dx \\ & \quad +4 \int r^{-s}\psi^2 \sum_{ij} a_{ij} \partial_i z \partial_j (r^{\tau/2}\phi) z A(r^{\tau/2}\phi) dx. \end{array} \tag{3.6}$$

With the choice of $\tau = s + \beta + 2$, we can compute

$$\begin{array}{rl} I & := \int r^{-s}\psi^2 \sum_{ij} a_{ij} \partial_i z \partial_j (r^{\tau/2}\phi) r^{\tau/2}\phi A z dx \\ & = \beta \int \sum_{ij} a_{ij} \partial_i z x_j A z dx + \tau/2 \int r^\beta \sum_{ij} a_{ij} \partial_i z x_j A z dx. \end{array}$$

It is readily seen that the leading term (for large β) of I is $\beta \int \sum_{ij} a_{ij} \partial_i z x_j A z dx$ provided r_0 is sufficiently small. Repeated integration by parts shows that

$$\begin{array}{rl} 2\int \sum_{ij} a_{ij} \partial_i z x_j A z dx & = 2 \int \sum_{ij} a_{ij} \partial_i z x_j \sum_{kl} a_{kl} \partial_k \partial_l z dx \\ & = -\int \sum_{ijkl} \partial_i z \partial_l (a_{kl} a_{ij} x_j) \partial_k z dx \\ & \quad +\int \sum_{ijkl} \partial_k z \partial_i (a_{kl} a_{ij} x_j) \partial_l z dx \\ & \quad -\int \sum_{ijkl} \partial_l z \partial_k (a_{kl} a_{ij} x_j) \partial_i z dx. \end{array} \tag{3.7}$$

Using (3.7) we obtain that

$$\begin{array}{l} |I| \le c\beta |\int \sum_{ijkl} \partial_i z \partial_l (a_{kl} a_{ij} x_j) \partial_k z dx| \\ \le c\beta \|\nabla z\|^2 \\ \le c\beta (\|\nabla (r^{-\tau/2}\psi) u\|^2 + \|r^{-\tau/2}\psi \nabla u\|^2) \\ \le c(\beta^3 \int r^{-s-3\beta-4}\psi^2 |u|^2 dx + \beta \int r^{-s-\beta-2}\psi^2 |\nabla u|^2 dx). \end{array} \tag{3.8}$$

In view of $\partial_j r^{\tau/2}=(\tau/2)r^{\tau/2-2}x_j$ and $\partial_j\phi=\beta r^{-\beta-2}x_j\phi$, then we can compute

$$\begin{aligned}J &:= \int r^{-s}\psi^2\sum_{ij}a_{ij}\partial_i z\partial_j(r^{\tau/2}\phi)zA(r^{\tau/2}\phi)dx\\ &= \beta\int r^{-s+\tau/2-\beta-2}\psi\sum_{ij}a_{ij}\partial_i zx_jzA(r^{\tau/2}\phi)dx\\ &\quad+\tau/2\int r^{-s+\tau/2-2}\psi\sum_{ij}a_{ij}\partial_i zx_jzA(r^{\tau/2}\phi)dx.\end{aligned}$$

Straightforward calculations show that

$$\partial_i\partial_j\phi=(\beta^2x_ix_jr^{-2\beta-4}+\beta\delta_{ij}r^{-\beta-2}-\beta(\beta+2)x_ix_jr^{-\beta-4})\phi$$

and

$$\partial_i\partial_j r^{\tau/2}=(\tau/2)(\tau/2-2)r^{\tau/2-4}x_ix_j+(\tau/2)r^{\tau/2-2}\delta_{ij}.$$

Hence, we can see that

$$\begin{aligned}&A(r^{\tau/2}\phi)=\sum_{ij}a_{ij}\partial_i\partial_j(r^{\tau/2}\phi)\\ &=\sum_{ij}\left(a_{ij}\phi\partial_i\partial_j(r^{\tau/2})+2a_{ij}(\partial_i r^{\tau/2}\partial_j\phi)+a_{ij}r^{\tau/2}\partial_i\partial_j\phi\right)\\ &=\sum_{ij}\Big(a_{ij}[(\tau/2)(\tau/2-2)r^{\tau/2-4}x_ix_j+(\tau/2)r^{\tau/2-2}\delta_{ij}]\phi+\tau\beta a_{ij}r^{\tau/2-\beta-4}x_ix_j\phi\\ &\quad+a_{ij}[\beta^2x_ix_jr^{-2\beta+\tau/2-4}+\beta\delta_{ij}r^{-\beta+\tau/2-2}-\beta(\beta+2)x_ix_jr^{-\beta+\tau/2-4}]\phi\Big).\end{aligned}$$

So the dominated term of J is

$$\beta^3\int r^{-2\beta-4}\sum_{ijkl}a_{ij}\partial_i zx_ja_{kl}x_kx_lzdx$$

provided β is sufficiently large and r_0 is sufficiently small. Note that we have chosen $\tau=s+\beta+2$. Integrating by parts and using the ellipticity condition (2.3), we can see that

$$\begin{aligned}&\beta^3\int r^{-2\beta-4}\sum_{ijkl}a_{ij}\partial_i zx_ja_{kl}x_kx_lzdx\\ &=-\tfrac{1}{2}\beta^3\int z\sum_{ijkl}\partial_i(r^{-2\beta-4}a_{ij}x_ja_{kl}x_kx_l)zdx\\ &\geq(1-o(\beta))\beta^4\int r^{-2\beta-6}\sum_{ijkl}a_{ij}x_ix_ja_{kl}x_kx_l|z|^2dx\\ &=(1-o(\beta))\beta^4\int r^{-s-3\beta-8}\psi^2\sum_{ijkl}a_{ij}x_ix_ja_{kl}x_kx_l|u|^2dx,\end{aligned}$$

where $0\leq o(\beta)\to 0$ as $\beta\to\infty$. In other words, we have that

$$J\geq(1-o(\beta))\beta^4\int r^{-s-3\beta-8}\psi^2\sum_{ijkl}a_{ij}x_ix_ja_{kl}x_kx_l|u|^2dx. \tag{3.9}$$

Combining (3.6), (3.8) and (3.9) gives

$$\begin{aligned}&\int r^{-s}\psi^2|Au|^2dx+c(\beta^3\int r^{-s-3\beta-4}\psi^2|u|^2dx+\beta\int r^{-s-\beta-2}\psi^2|\nabla u|^2dx)\\ &\geq 4(1-o(\beta))\beta^4\int r^{-s-3\beta-8}\psi^2\sum_{ijkl}a_{ij}x_ix_ja_{kl}x_kx_l|u|^2dx\end{aligned}$$

from which we can derive

$$\begin{aligned}&\int r^{-s}\psi^2|Au|^2dx+c\beta\int r^{-s-\beta-2}\psi^2|\nabla u|^2dx\\ &\geq 3(1-o(\beta))\beta^4\int r^{-s-3\beta-8}\psi^2\sum_{ijkl}a_{ij}x_ix_ja_{kl}x_kx_l|u|^2dx\end{aligned} \tag{3.10}$$

using the ellipticity condition (2.3).

Again integrating by parts, we conclude

$$\begin{aligned}&\int r^{-s-\beta-4}\psi^2\sum_{ijkl}a_{kl}x_kx_la_{ij}\partial_i u\partial_j udx\\ &\leq|\int u\partial_i(r^{-s-\beta-4}\psi^2)\sum_{ijkl}a_{kl}x_kx_la_{ij}\partial_j udx|\\ &\quad+|\int r^{-s-\beta-4}\psi^2u\sum_{ijkl}\partial_i(a_{kl}x_kx_la_{ij})\partial_j udx|\\ &\quad+|\int r^{-s-\beta-4}\psi^2u\sum_{ijkl}a_{kl}x_kx_la_{ij}\partial_i\partial_j udx|\\ &:=K_1+K_2+K_3.\end{aligned} \tag{3.11}$$

We first handle the term K_1. If we define the inner product $\langle X, Y\rangle = \sum_{ij} a_{ij} X_i Y_j$ for vectors X and Y, the following Cauchy-Schwarz inequality obviously holds

$$|\langle X, Y\rangle| \le |\langle X, X\rangle|^{1/2} |\langle Y, Y\rangle|^{1/2}. \tag{3.12}$$

Note that

$$\partial_i(r^{-s-\beta-4}\psi^2) = (-s-\beta-4) r^{-s-\beta-6} x_i \psi^2 - 2\beta r^{-s-2\beta-6} x_i \psi^2. \tag{3.13}$$

Thus, using (3.12) and (3.13), we can estimate

$$\begin{aligned} K_1 &= |\textstyle\int u\partial_i(r^{-s-\beta-4}\psi^2)\sum_{ijkl} a_{kl}x_kx_la_{ij}\partial_j u dx| \\ &\le \textstyle\int (2+o(\beta))\beta r^{-s-2\beta-6}\sum_{kl} a_{kl}x_kx_l\psi^2|u||\sum_{ij} a_{ij}x_i\partial_j u|dx \\ &\le \textstyle\int (2+o(\beta))\beta r^{-s-2\beta-6}\sum_{kl} a_{kl}x_kx_l\psi^2|u||\sum_{ij} a_{ij}x_ix_j|^{1/2}|\sum_{ij} a_{ij}\partial_i u\partial_j u|^{1/2}dx \\ &\le (1/2)(2+o(\beta))^2\beta^2 \textstyle\int r^{-s-3\beta-8}\sum_{ijkl} a_{kl}x_kx_la_{ij}x_ix_j\psi^2|u|^2dx \\ &\quad +(1/2)\textstyle\int r^{-s-\beta-4}\psi^2\sum_{ijkl} a_{kl}x_kx_la_{ij}\partial_i u\partial_j u dx. \end{aligned} \tag{3.14}$$

Here we have used the relation $|ab| \le (a^2+b^2)/2$. For K_2 and K_3, straightforward computations give that

$$K_2 \le c(r_0^\beta \int r^{-s-3\beta-4}\psi^2|u|^2dx + r_0^{\beta+2}\int r^{-s-\beta-2}\psi^2|\nabla u|^2dx) \tag{3.15}$$

and

$$K_3 \le c(r_0^\beta\beta^2 \int r^{-s-3\beta-4}\psi^2|u|^2dx + \beta^{-2}\int r^{-s}\psi^2|Au|^2dx) \tag{3.16}$$

provided $r_0 \ll 1$, where the constant c only depends on the coefficients a_{ij}'s. Plugging (3.14), (3.15) and (3.16) into (3.11) and multiplying the new inequality by β^2, we obtain that

$$\begin{aligned} &\beta^2\textstyle\int r^{-s-\beta-4}\psi^2\sum_{ijkl} a_{kl}x_kx_la_{ij}\partial_i u\partial_j u dx \\ &\le (2+o(\beta))\beta^4\textstyle\int r^{-s-3\beta-8}\sum_{ijkl} a_{kl}x_kx_la_{ij}x_ix_j\psi^2|u|^2dx \\ &\quad +(1/2)\beta^2\textstyle\int r^{-s-\beta-4}\psi^2\sum_{ijkl} a_{kl}x_kx_la_{ij}\partial_i u\partial_j u dx \\ &\quad +c(r_0^\beta\beta^2\textstyle\int r^{-s-3\beta-4}\psi^2|u|^2dx + r_0^{\beta+2}\beta^2\int r^{-s-\beta-2}\psi^2|\nabla u|^2dx) \\ &\quad +c(r_0^\beta\beta^4\textstyle\int r^{-s-3\beta-4}\psi^2|u|^2dx + \int r^{-s}\psi^2|Au|^2dx). \end{aligned} \tag{3.17}$$

Adding (3.17) to (3.10) immediately yields

$$\begin{aligned} &3(1-o(\beta))\beta^4\textstyle\int r^{-s-3\beta-8}\psi^2\sum_{ijkl} a_{ij}x_ix_ja_{kl}x_kx_l|u|^2dx \\ &+\beta^2\textstyle\int r^{-s-\beta-4}\psi^2\sum_{ijkl} a_{kl}x_kx_la_{ij}\partial_i u\partial_j u dx \\ &\le (2+o(\beta))\beta^4\textstyle\int r^{-s-3\beta-8}\sum_{ijkl} a_{kl}x_kx_la_{ij}x_ix_j\psi^2|u|^2dx \\ &\quad +(1/2)\beta^2\textstyle\int r^{-s-\beta-4}\psi^2\sum_{ijkl} a_{kl}x_kx_la_{ij}\partial_i u\partial_j u dx + c\beta\int r^{-s-\beta-2}\psi^2|\nabla u|^2dx \\ &\quad +c(r_0^\beta\beta^2\textstyle\int r^{-s-3\beta-4}\psi^2|u|^2dx + r_0^{\beta+2}\beta^2\int r^{-s-\beta-2}\psi^2|\nabla u|^2dx) \\ &\quad +c(r_0^\beta\beta^4\textstyle\int r^{-s-3\beta-4}\psi^2|u|^2dx + \int r^{-s}\psi^2|Au|^2dx). \end{aligned} \tag{3.18}$$

Taking β sufficiently large in (3.18) and using the ellipticity condition (2.3), we now conclude that

$$\beta^4\int r^{-s-3\beta-4}\psi^2|u|^2dx + \beta^2\int r^{-s-\beta-2}\psi^2|\nabla u|^2dx \le c\int r^{-s}\psi^2|Au|^2dx$$

which immediately implies (2.5).

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