Math 300 **Introduction to Mathematical Reasoning** **Autumn 2017**

Axioms for the Real Numbers (CORRECTED)

PRIMITIVE TERMS

To avoid circularity, we cannot give every term a rigorous mathematical definition; we have to accept some things as undefined terms. For this course, we will take the following fundamental notions as primitive undefined terms. You already know intuitively what these terms mean; but the only facts about them that can be used in proofs are the ones expressed in the axioms listed below (and any theorems that can be proved from the axioms).

- ***Real number***
- ***Addition***
- ***Multiplication***
- ***Positive***
- ***Integer***

These terms are assumed to satisfy certain axioms. The axioms come in four groups.

GROUP I: FIELD AXIOMS

We assume that there exists a set $\mathbb{R}$, whose elements are called ***real numbers***, endowed with two binary operations called ***addition*** (denoted by $a + b$) and ***multiplication*** (denoted by ab or $a \cdot b$ or $a \times b$). The first group of axioms concerns basic algebraic properties of these operations.

We assume the following axioms about $\mathbb{R}$:

Axiom 1. (Closure of $\mathbb{R}$) If a and b are real numbers, then so are $a + b$ and ab.
Axiom 2. (Commutativity) $a + b = b + a$ and $ab = ba$ for all real numbers a and b.
Axiom 3. (Associativity) $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ for all real numbers a, b, and c.
Axiom 4. (Distributivity) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all real numbers a, b, and c.
Axiom 5. (Identities) There exist two distinct real numbers, denoted by 0 and 1, such $0 + a = a + 0 = a$ and $1 \cdot a = a \cdot 1 = a$ for every real number a.
Axiom 6. (Additive Inverses) For every real number a, there exists a real number $-a$, called the ***additive inverse*** or ***opposite*** of a, such that $a + (-a) = (-a) + a = 0$.
Axiom 7. (Multiplicative Inverses) For every nonzero real number a, there exists a real number a^{-1}, called the ***multiplicative inverse*** or ***reciprocal*** of a, such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Any set endowed with two operations satisfying these axioms is called a ***field***, so our assumptions so far can be summarized by saying that $\mathbb{R}$ is a field. (Can you think of any other fields?)

Based on these axioms, we can begin to prove some theorems. But first, some definitions. In all of these definitions, a and b represent arbitrary real numbers.

Definitions

- The numbers **2** through **10** are defined by $2 = 1+1$, $3 = 2+1$, etc. The decimal representations for other numbers are defined by the usual rules of decimal notation: For example, 23 is defined

to be $2 \cdot 10 + 3$, etc.

- The ***difference between a and b***, denoted by $\boldsymbol{a - b}$, is the real number defined by $a - b = a + (-b)$, and is said to be obtained by ***subtracting b from a***.
- If $b \neq 0$, the ***quotient of a and b***, denoted by $\boldsymbol{a/b}$, is the real number defined by $a/b = ab^{-1}$, and is said to be obtained by ***dividing a by b***.
- The ***square of a***, denoted by $\boldsymbol{a^2}$, is the real number $a \cdot a$.

A Note About Equality

Before we begin proving theorems, we need to mention the role played by *equality* in mathematical reasoning. Equality is the most fundamental relation in all of mathematics. It can be used between any two mathematical objects of the same type, such as numbers, matrices, ordered pairs, sets, functions, etc. To say that $a = b$ is simply to say that the symbols a and b represent the very same object. No matter what type of objects it is applied to, equality always has the following fundamental properties.

- (Reflexivity) $a = a$.
- (Symmetry) If $a = b$, then $b = a$.
- (Transitivity) If $a = b$ and $b = c$, then $a = c$.
- (Substitution) If $a = b$, then b may be substituted for any or all occurrences of a in any mathematical statement without affecting that statement's truth value.

All of the familiar rules for "doing the same thing to both sides of an equation" are really just applications of these properties. Here is common example of this type of rule:

- Suppose a, b, c are any real numbers. If $a = b$, then $a + c = b + c$.

To verify this, just note that the reflexive property implies $a + c = a + c$, and then substituting b for a on the right-hand side (but not the left) leads to $a + c = b + c$. You can probably think of many other rules like this, all of which can be proved by variations on the same argument.

Theorems Based on the Field Axioms

The following theorems can be proved from the axioms in the order listed below. In all of these theorems, a, b, c, d represent arbitrary real numbers. (In other words, each of these theorem statements should be read as if it started with "For all real numbers $a, b, c, d, \ldots$.")

Theorem 1. (Cancellation Laws) If $a + c = b + c$, then $a = b$. If $ac = bc$ and $c \neq 0$, then $a = b$.

Theorem 2. (Uniqueness of Identities) The numbers 0 and 1 in the identity axiom are unique.

Theorem 3. (Uniqueness of Inverses) For any a, the number $-a$ such that $a + (-a) = (-a) + a = 0$ is unique; and for any nonzero a, the number a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$ is unique.

Theorem 4. $-0 = 0$.

Theorem 5. $a - a = 0$.

Theorem 6. $a - b = 0$ if and only if $a = b$.

Theorem 7. $a - 0 = a$.

Theorem 8. $0 - a = -a$.

Theorem 9. $-(-a) = a$.
Theorem 10. $0a = 0$.
Theorem 11. $-a = (-1)a$.
Theorem 12. $(-a)b = -(ab)$ and $a(-b) = -(ab)$.
Theorem 13. $(-a)(-b) = ab$.
Theorem 14. $a(b-c) = ab - ac$ and $(b-c)a = ba - ca$.
Theorem 15. $-(a+b) = -a - b$.
Theorem 16. $-(a-b) = b - a$.
Theorem 17. $(a+b)(c+d) = ac + ad + bc + bd$.
Theorem 18. $(a-b)(c+d) = (c+d)(a-b) = ac + ad - bc - bd$.
Theorem 19. $(a-b)(c-d) = ac - ad - bc + bd$.
Theorem 20. $1^{-1} = 1$.
Theorem 21. $a/1 = a$.
Theorem 22. If $a \neq 0$, then $a^{-1} \neq 0$.
Theorem 23. If $a \neq 0$, then $a/a = 1$.
Theorem 24. If $a \neq 0$, then $a^{-1} = 1/a$.
Theorem 25. If $a \neq 0$, then $(-a)^{-1} = -(1/a)$.
Theorem 26. If $a \neq 0$, then $\left(a^{-1}\right)^{-1} = a$.
Theorem 27. If $ab = 0$, then $a = 0$ or $b = 0$.
Theorem 28. If $a \neq 0$ and $b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$.
Theorem 29. If $a \neq 0$ and $b \neq 0$, then $(a/b)^{-1} = b/a$.
Theorem 30. If $a^2 = b^2$, then $a = \pm b$.
Theorem 31. $a^2 = 0$ if and only if $a = 0$.
Theorem 32. $(-a)^2 = a^2$.
Theorem 33. $(a^{-1})^2 = 1/a^2$.
Theorem 34. If $b \neq 0$ and $d \neq 0$, then $(a/b)(c/d) = (ac)/(bd)$.
Theorem 35. If $b \neq 0$, $c \neq 0$, and $d \neq 0$, then $(a/b)/(c/d) = (ad)/(bc)$.
Theorem 36. If $c \neq 0$, then $(a/c) + (b/c) = (a+b)/c$.
Theorem 37. If $b \neq 0$ and $d \neq 0$, then $(a/b) + (c/d) = (ad + bc)/(bd)$.
Theorem 38. If $b \neq 0$ and $c \neq 0$, then $(ac)/(bc) = a/b$.

GROUP II: ORDER AXIOMS

Our second group of axioms will allow us to talk about concepts like *positive*, *negative*, *greater than*, and *less than*. For this group, we assume that there exists a subset $\mathbb{R}^+ \subseteq \mathbb{R}$ whose elements are called ***positive real numbers***, such that the following statements are true.

Axiom 8. (Closure of $\mathbb{R}^+$) If a and b are positive real numbers, then so are $a + b$ and ab.
Axiom 9. (Trichotomy Axiom) If a is a real number, then one and only one of the following three statements is true: $a \in \mathbb{R}^+$, $-a \in \mathbb{R}^+$, or $a = 0$.

A field $\mathbb{R}$ together with a subset $\mathbb{R}^+$ satisfying these two axioms is called an ***ordered field***. So our axioms so far state that $\mathbb{R}$ is an ordered field.

More Definitions

- ***a is less than b***, denoted by $\boldsymbol{a < b}$, means $b - a$ is positive.
- ***a is less than or equal to b***, denoted by $\boldsymbol{a \leq b}$, means $a < b$ or $a = b$.
- ***a is greater than b***, denoted by $\boldsymbol{a > b}$, means $b < a$.

- ***a is greater than or equal to b***, denoted by $\boldsymbol{a \geq b}$, means $a > b$ or $a = b$.
- A real number a is ***negative*** if $a < 0$.
- A real number a is ***nonnegative*** if $a \geq 0$.
- A real number a is ***nonpositive*** if $a \leq 0$.
- If S is a set of real numbers, a real number b is said to be the ***largest element of S*** or the ***maximum of S*** if b is an element of S and, in addition, $b \geq x$ whenever x is any element of S. The terms ***smallest element*** and ***minimum*** are defined similarly.
- The ***absolute value of a*** is the number $|a|$ defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

More Theorems

Here are theorems about ordered fields. As before, these can be proved from the axioms in the order listed below. In all of these theorems, a, b, c, d represent arbitrary real numbers.

Theorem 39. a is positive if and only if $a > 0$.
Theorem 40. (Trichotomy Law for Inequalities) If a and b are real numbers, then one and only one of the following three statements is true: $a < b$, $a = b$, or $a > b$.
Theorem 41. (Transitive Law for Inequalities) If $a < b$ and $b < c$, then $a < c$.
Theorem 42. If $a \leq b$ and $b < c$, then $a < c$.
Theorem 43. If $a < b$ and $b \leq c$, then $a < c$.
Theorem 44. If $a \leq b$ and $b \leq c$, then $a \leq c$.
Theorem 45. If $a < b$, then $a + c < b + c$.
Theorem 46. If $a \leq b$, then $a + c \leq b + c$.
Theorem 47. If $a < c$ and $b < d$, then $a + b < c + d$.
Theorem 48. If $a \leq c$ and $b < d$, then $a + b < c + d$.
Theorem 49. If $a < c$ and $b \leq d$, then $a + b < c + d$.
Theorem 50. If $a \leq c$ and $b \leq d$, then $a + b \leq c + d$.
Theorem 51. If $a < b$ and $c > 0$, then $ac < bc$.
Theorem 52. If $a \leq b$ and $c \geq 0$, then $ac \leq bc$.
Theorem 53. If $a < b$ and $c < 0$, then $ac > bc$.
Theorem 54. If $a \leq b$ and $c \leq 0$, then $ac \geq bc$.
Theorem 55. $a^2 > 0$ if and only if $a \neq 0$.
Theorem 56. $1 > 0$.
Theorem 57. If $a < b$ and a and b are both positive, then $a^2 < b^2$.
Theorem 58. If $a \leq b$ and a and b are both nonnegative, then $a^2 \leq b^2$.
Theorem 59. If $a < b$ and a and b are both negative, then $a^2 > b^2$.
Theorem 60. If $a \leq b$ and a and b are both nonpositive, then $a^2 \geq b^2$.
Theorem 61. If $a < b$, then $-a > -b$.
Theorem 62. If $a \leq b$, then $-a \geq -b$.
Theorem 63. $a > 0$ if and only if $-a < 0$.
Theorem 64. $ab > 0$ if and only if a and b are both positive or both negative.
Theorem 65. $ab < 0$ if and only if one is positive and the other is negative.
Theorem 66. $|a| = 0$ if and only if $a = 0$.
Theorem 67. $|a| > 0$ if and only if $a \neq 0$.
Theorem 68. $|a| \geq 0$.
Theorem 69. $a \leq |a|$.
Theorem 70. $|-a| = |a|$.

Theorem 71. $|a| = \max\{a, -a\}$.
Theorem 72. $|a^{-1}| = 1/|a|$ if $a \neq 0$.
Theorem 73. $|ab| = |a|\,|b|$.
Theorem 74. $|a/b| = |a|/|b|$ if $b \neq 0$.
Theorem 75. If $|a| = |b|$, then $a = \pm b$.
Theorem 76. If a and b are both nonnegative, then $|a| \geq |b|$ if and only if $a \geq b$.
Theorem 77. If a and b are both negative, then $|a| \geq |b|$ if and only if $a \leq b$.
Theorem 78. $|a| < b$ if and only if $a > -b$ and $a < b$.
Theorem 79. $|a| \leq b$ if and only if $a \geq -b$ and $a \leq b$.
Theorem 80. $|a| > b$ if and only if $a < -b$ or $a > b$.
Theorem 81. $|a| \geq b$ if and only if $a \leq -b$ or $a \geq b$.
Theorem 82. (The triangle inequality) $|a + b| \leq |a| + |b|$.
Theorem 83. (The reverse triangle inequality) $\big||a| - |b|\big| \leq |a - b|$.
Theorem 84. (Density of real numbers) If $a < b$, there exists a real number c such that $a < c < b$.
Theorem 85. There does not exist a smallest or largest real number.
Theorem 86. There does not exist a smallest positive real number.

GROUP III: AXIOMS ABOUT THE INTEGERS

So far, we have not mentioned the integers. It is possible, with some extra effort, to *define* the integers as a certain subset of the real numbers, and to *prove* all of the properties we need to know about them. But doing so would take us too far afield, so it is more efficient just to treat "integer" as a primitive undefined term like "real number" or "positive," and introduce a few additional axioms that establish the properties we need.

Thus we assume there is a subset $\mathbb{Z} \subseteq \mathbb{R}$ whose elements are called ***integers***, satisfying the following axioms.

Axiom 10. 1 is an integer.
Axiom 11. (Closure of $\mathbb{Z}$) If a and b are integers, then so are $a + b$, $a - b$, and ab.
Axiom 12. (The Well-Ordering Axiom) Every nonempty set of positive integers contains a smallest integer.

Some Theorems About Integers

Theorem 87. 0 is an integer.
Theorem 88. 1 is the smallest positive integer.
Theorem 89. If a is an integer, then so is $-a$.
Theorem 90. If a and b are integers such that $a > b$, then $a \geq b + 1$.
Theorem 91. There does not exist a smallest or largest integer.

GROUP IV: LEAST UPPER BOUND AXIOM

We have one more axiom to go, one that is somewhat more subtle than the rest. We won't use it very much in this course, but it will be extremely important in later courses because it is the foundation on which calculus is built. Before we state it, we need a few more definitions.

More Definitions

- If S is a set of real numbers, a real number b (not necessarily in S) is said to be an ***upper bound for S*** if $b \geq x$ for every x in S. A ***lower bound*** is defined similarly.
- If S is a set of real numbers, a real number b (not necessarily in S) is said to be a ***least upper bound for S*** if it is an upper bound, and in addition every other upper bound b' for S satisfies $b' \geq b$. A ***greatest lower bound*** is defined similarly.

Axiom 13. (The least upper bound axiom) Every nonempty set of real numbers that has an upper bound has a least upper bound.

An ordered field satisfying the least upper bound axiom is said to be ***complete***. So now we can summarize our entire set of assumptions about the real numbers:

There exists a complete ordered field $\mathbb{R}$***.***

Some Theorems That Use the Least Upper Bound Axiom

Theorem 92. If a is any nonnegative real number, there is a unique nonnegative real number $\sqrt{a}$, called the ***square root of a***, such that $\left(\sqrt{a}\right)^2 = a$.
Theorem 93. If $a^2 = b$, then $a = \pm\sqrt{b}$.
Theorem 94. $\sqrt{a^2} = |a|$.
Theorem 95. If $a < b$ and a and b are both nonnegative, then $\sqrt{a} < \sqrt{b}$.

RATIONAL NUMBERS

Rational numbers are those that can be expressed as fractions with integral numerator and denominator. We don't need any new axioms about rational numbers; just a definition and a few theorems.

Definition

- A real number a is said to be ***rational*** if there are integers p and q with $q \neq 0$ such that $a = p/q$. It is said to be ***irrational*** if it is not rational. The set of all rational numbers is denoted by $\mathbb{Q}$.

Theorems About Rational Numbers

Theorem 96. Every integer is a rational number.
Theorem 97. Closure of $\mathbb{Q}$: If a and b are rational numbers, so are ab, $a + b$, and $a - b$. If in addition $b \neq 0$, then a/b is also rational.
Theorem 98. If a is rational, then so is $-a$. If in addition $a \neq 0$, then a^{-1} is also rational.
Theorem 99. $\sqrt{2}$ is irrational.