Math 300 **Introduction to Mathematical Reasoning** **Fall 2017**

More About Countable Sets

Please read this handout after Section 9.2 in the textbook.

Theorems about Countable Sets

This handout summarizes some of the most important results about countable sets. Many of these are proved either in the textbook or in its exercises, but I want to bring these properties together in a way that shows their similarity with properties of finite sets.

First, we need some lemmas.

Lemma 1. *For every natural number n, the set $\mathbb{N} \times \mathbb{N}_n$ is countably infinite.*

Proof. We'll prove this by induction on n. When $n = 1$, we have $\mathbb{N} \times \mathbb{N}_1 = \mathbb{N} \times \{1\}$, and it follows from Exercise 2 on page 460 that this set is equivalent to $\mathbb{N}$.

For the inductive step, let $k \in \mathbb{N}$ and assume that $\mathbb{N} \times \mathbb{N}_k$ is countably infinite. Note that

$$\mathbb{N} \times \mathbb{N}_{k+1} = (\mathbb{N} \times \mathbb{N}_k) \cup (\mathbb{N} \times \{k+1\}),$$

and the sets on the right-hand side are disjoint because no element of the first set has $k+1$ as its second coordinate, while every element of the second set does. The first set is countably infinite by the inductive hypothesis, and the second by Exercise 2 on page 460. Therefore, $\mathbb{N} \times \mathbb{N}_{k+1}$ is a union of two disjoint countably infinite sets, so it follows from Theorem 9.17 that it is countably infinite. $\square$

Lemma 2. *Every natural number can be expressed in the form $n = 2^p q$, where p is a nonnegative integer and q is an odd natural number.*

Proof. We will prove this by strong induction. For the base case $n = 1$, just note that $n = 2^0 \cdot 1$.

Now let $k \in \mathbb{N}$, and suppose that every natural number less than k can be written in the desired form. If k is odd, we just write $k = 2^0 k$. If k is even, then there is an integer l such that $k = 2l$, and l is positive because k is. Since $l < k$, the inductive hypothesis implies that there exist a nonnegative integer p and an odd natural number q such that $l = 2^p q$, and then $k = 2l = 2^{p+1} q$, which satisfies the conclusion. $\square$

Lemma 3. *The set $\mathbb{N} \times \mathbb{N}$ is countably infinite.*

Proof. (This is Exercise 9 on page 474 of the text.) Define a function $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(m,n) = 2^{m-1}(2n-1)$. To show that f is injective, suppose (m_1, n_1), (m_2, n_2) are elements of $\mathbb{N} \times \mathbb{N}$ such that $f(m_1, n_1) = f(m_2, n_2)$, which is to say

$$2^{m_1 - 1}(2n_1 - 1) = 2^{m_2 - 1}(2n_2 - 1). \tag{1}$$

We will first prove by contradiction that $m_1 = m_2$. Suppose not; then one is larger, and we may assume without loss of generality that $m_2 > m_1$. Multiplying both sides of (1) by 2^{1-m_1}, we obtain

$$2n_1 - 1 = 2^{m_2 - m_1}(2n_2 - 1).$$

The fact that $m_2 > m_1$ implies that the right-hand side is even, while the left-hand side is odd; this is a contradiction, so we can conclude that $m_1 = m_2$. Then simple algebra shows that $n_1 = n_2$ as well, so $(m_1, n_1) = (m_2, n_2)$.

To prove surjectivity, let $x \in \mathbb{N}$ be arbitrary. Lemma 2 shows that we can write $x = 2^p q$ for some nonnegative integer p and some odd natural number q. The fact that q is odd means that $q = 2j+1$ for some integer j, and the fact that $q \geq 1$ means $j \geq 0$. Therefore, $(p+1, j+1) \in \mathbb{N} \times \mathbb{N}$, and

$$f(p+1, j+1) = 2^{(p+1)-1}(2(j+1) - 1) = 2^p(2j+1) = 2^p q = x.$$

Thus we have shown that f is bijective, so $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$. □

Here is the main theorem of this handout.

Theorem 4 (Fundamental Properties of Countable Sets)**.** *Suppose A and B are countable sets.*

(*a*) *Every subset of A is countable.*

(*b*) *$A \cup B$ is countable.*

(*c*) *$A \times B$ is countable.*

Proof. Part (a) is Corollary 9.20 in the textbook. For the proof of part (b), see Exercise 1 at the end of this handout.

Here is a proof of (c). If A or B is empty, then $A \times B$ is empty, so we may assume both are nonempty. If A and B are both finite, then $A \times B$ is finite by Theorem 3 on the finite sets handout, and thus it is countable.

Next, suppose A is countably infinite and B is finite; thus $A \approx \mathbb{N}$ and $B \approx \mathbb{N}_n$ for some n. It follows from Exercise 7(a) on page 460 that $A \times B \approx \mathbb{N} \times \mathbb{N}_n$, which is countably infinite by Lemma 1 above. On the other hand, if A is finite and B is countably infinite, the preceding argument shows that $B \times A$ is countably infinite; the function $g(a, b) = (b, a)$ is a bijection from $A \times B$ to $B \times A$, so $A \times B$ is countably infinite also.

Finally, if A and B are both countably infinite, then Exercise 7(a) on page 460 shows that $A \times B \approx \mathbb{N} \times \mathbb{N}$, which is countably infinite by Lemma 3 above. □

Just as for finite sets, we have the following shortcuts for determining that a set is countable.

Theorem 5. *Let A be a nonempty set.*

(*a*) *If there exists an injection from A to a countable set, then A is countable.*

(*b*) *If there exists a surjection from a countable set to A, then A is countable.*

Proof. First we prove (a). Suppose B is countable and there exists an injection $f\colon A \to B$. Just as in the proof of Theorem 4 on the finite sets handout, we can define a bijection $f'\colon A \to f(A)$ by setting $f'(x) = f(x)$ for every $x \in A$. Since $f(A)$ is a subset of the countable set B, it is countable, and therefore so is A.

Now to prove part (b), suppose B is countable and there exists a surjection $f\colon B \to A$. First consider the case in which $B = \mathbb{N}$. Using the same argument as in Exercise 9 on page 461 of the textbook, we can construct an injective function $g\colon A \to \mathbb{N}$ by letting $g(a)$ be the smallest element of the set $f^{-1}(\{a\})$. It follows that A is countable by the result in part (a).

Now suppose B is an arbitrary countable set and there exists a surjection $f\colon B \to A$. The fact that A is nonempty guarantees that B is also nonempty. If B is finite, then it follows from Theorem 4 on the finite sets handout that A is finite, and hence countable. On the other hand, if B is countably infinite, there exists a bijection $g\colon \mathbb{N} \to B$. Then $f \circ g\colon \mathbb{N} \to A$ is a composition of surjections and hence surjective, so the argument in the preceding paragraph shows that A is countable. □

The textbook sketched an argument that the set of positive rational numbers is countably infinite (see Theorem 9.14). That argument is intuitively appealing, but it is notoriously hard to make precise, because there is no good formula for the bijection being described from $\mathbb{N}$ to $\mathbb{Q}^+$. Here is a proof that is perhaps less intuitive, but much more straightforwardly rigorous.

Theorem 6. *The set of positive rational numbers is countably infinite.*

Proof. Because $\mathbb{Q}^+$ contains the natural numbers, it is infinite, so we need only show it is countable. Define $g\colon \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ by $g(m, n) = m/n$. Since every positive rational number can be written as a quotient of positive integers, g is surjective. Since $\mathbb{N} \times \mathbb{N}$ is countable, it follows from Theorem 5(b) above that $\mathbb{Q}^+$ is countable. □

Exercise

1. Prove part (b) of Theorem 4. [Hint: Use Theorems 9.16 and 9.17 in the textbook, and argue as you did in Exercise 1 on the finite sets handout.]