Math 300 **Introduction to Mathematical Reasoning** **Autumn 2017**

Mathematical Induction (CORRECTED)

Chapter 4 in our textbook describes the technique of proof by mathematical induction. The essential facts that make proofs by induction work are stated in the book as *The Principle of Mathematical Induction* (page 173) and *The Extended Principle of Mathematical Induction* (page 190). Here are the statements:

Theorem 1 (The Principle of Mathematical Induction)**.** *If T is a subset of $\mathbb{N}$ such that*

1. $1 \in T$, *and*

2. *For every $k \in \mathbb{N}$, if $k \in T$, then $(k+1) \in T$,*

then $T = \mathbb{N}$.

Theorem 2 (The Extended Principle of Mathematical Induction)**.** *Let M be an integer. If T is a subset of $\mathbb{Z}$ such that*

1. $M \in T$, *and*

2. *For every $k \in \mathbb{Z}$ with $k \geq M$, if $k \in T$, then $(k+1) \in T$,*

then T contains all integers greater than or equal to M. That is, $\{n \in \mathbb{Z} \mid n \geq M\} \subseteq T$.

The book contains many examples of how these principles are used. Unfortunately, though, the author doesn't prove that they're valid – instead, he accepts them as more "axioms."

In order to avoid piling axioms on top of axioms, it is preferable (and not very hard) to prove the principles of induction from the axioms we already have. Here is a proof of the extended principle; the ordinary principle follows easily from this one, simply by taking $M = 1$.

Proof of the Extended Principle of Mathematical Induction. Let T be a subset of $\mathbb{Z}$ satisfying the hypotheses. We are given that $M \in T$, so what we have to prove is that $M+1 \in T$, $M+2 \in T$, etc.; in other words, we need to prove

$$\text{For every positive integer } n,\ M + n \in T.$$

Assume for the sake of contradiction that this is not the case. This means that there is some positive integer n such that $M + n \notin T$. Let S be the set consisting of all positive integers n such that $M + n \notin T$; our hypothesis guarantees that $S \neq \emptyset$. Therefore, by the Well-Ordering Axiom, there must be a *smallest* positive integer in S; call it $n_{\min}$. In other words, $n_{\min}$ is the smallest positive integer such that $M + n_{\min} \notin T$.

Now hypotheses 1 and 2 together guarantee that $M + 1 \in T$; therefore $n_{\min} \neq 1$. Let $n_{\text{smaller}} = n_{\min} - 1$. Since $n_{\min}$ is a positive integer not equal to 1, we have $n_{\min} > 1$ and therefore $n_{\text{smaller}} > 0$. Since n_{smaller} is a positive integer less than $n_{\min}$, it follows that $n_{\text{smaller}} \notin S$, which means that $M + n_{\text{smaller}} \in T$. But then hypothesis 2 shows that $M + n_{\text{smaller}} + 1 \in T$. Since $n_{\text{smaller}} + 1 = n_{\min}$, we have shown $M + n_{\min} \in T$, which is a contradiction. □

The other variation on mathematical induction is a principle that the book refers to as *The Second Principle of Mathematical Induction* (page 193). Since just about everyone else in the world refers to it as *Strong Induction*, that's what we will call it.

Theorem 3 (The Principle of Strong Induction)**.** *Let M be an integer. If T is a subset of $\mathbb{Z}$ such that*

1. $M \in T$, *and*
2. *For every $k \in \mathbb{Z}$ with $k \geq M$, if $\{M, M+1, \ldots, k\} \subseteq T$, then $(k+1) \in T$,*

then T contains all integers greater than or equal to M. That is, $\{n \in \mathbb{Z} \mid n \geq M\} \subseteq T$.

Proof. Let $T \subseteq \mathbb{Z}$ be a subset satisfying the hypotheses, and assume for the sake of contradiction that T does not contain all the integers greater than or equal to M. Then by the same reasoning as in the proof of Theorem 2, there is a smallest positive integer $n_{\min}$ such that $M + n_{\min} \notin T$. The fact that $n_{\min}$ is the smallest such positive integer means that the integers $M, M+1, \ldots, M+n_{\min}-1$ are all in T. But then hypothesis 2 implies that $M + n_{\min} \in T$, which is a contradiction. $\square$