A Tale of Knots & Games

Allison Henrich, Ph.D.

Seattle University

April 27, 2014

What is a knot?

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(We'll come back to this.)

Ancient knots in art

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Knot theory

2200 B.C. – Lerna

Figure; A seal-impression from the House of the Tiles in Lerna.

...knots have been appearing in art since at least 2200 BC.

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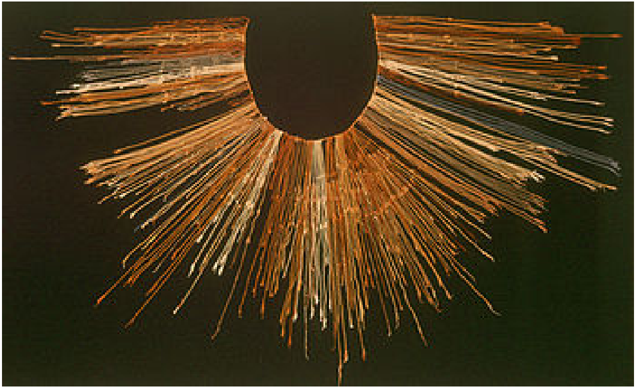
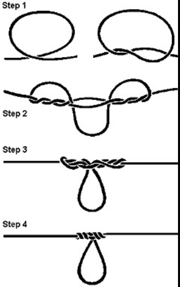
In 330 BC, Alexander the Great famously tried to untie the knot.

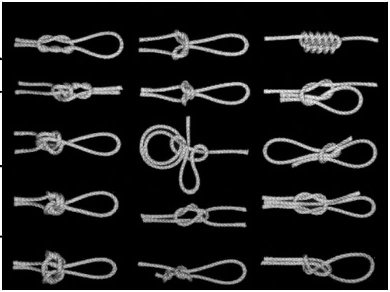
Upon failing to solve the puzzle the "correct" way, he unsheathed his sword and sliced the knot in half!

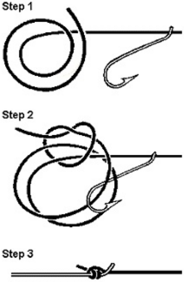
The Inca empire in fourteenth century South America used knots (quipu) for accounting.



Out to sea!







Knots have been put to use for fishing and sailing for as long as we can remember.

Knots and chemistry

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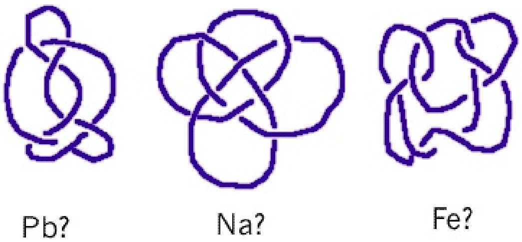
- Lord Kelvin and a scientist named Peter Tait set out to classify knots. This classification was meant to aid in the classification of atoms.

Knots and chemistry (cont.)



Recent history

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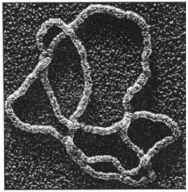
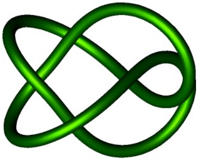
2. Chemistry (properties of molecules)
3. Biology (DNA replication)



What is a knot?

Definition (A Mathematical Notion)

A **knot** is a circle that doesn't intersect itself sitting in space.



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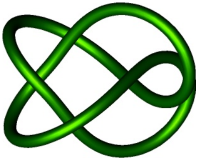

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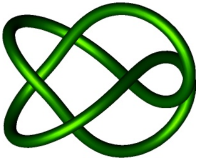
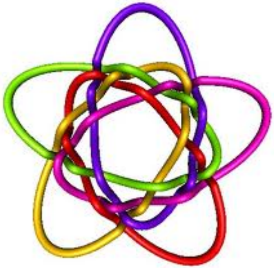


Intuitively, we say that two knots are **equivalent** if we can get from one to the other by bending, stretching, and rotating *as long as we don't break or cut* the knot anywhere. (Sorry, Alexander. No swords allowed!)

What are links?

Definition (A Mathematical Notion)

A **link** is a collection of non-intersecting knots (perhaps linked with one another) sitting in space.



Trivialities

- A trivial knot is called the **unknot**.

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- A trivial link is called the **unlink**.

Knot diagrams

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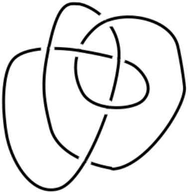

- A **knot diagram** is a closed curve in the plane containing crossings (no tangencies or triple-points!). We decorate these crossings in a particular way to indicate which is the over-strand and which is the under-strand of the crossing.

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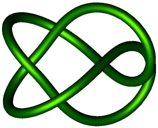

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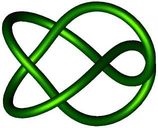
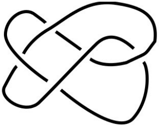


from **our viewpoint** ...



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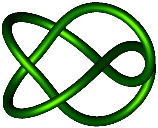

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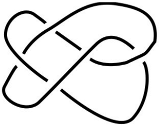

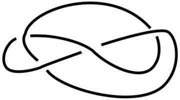

...or from a **“bird’s eye” viewpoint**.

Knot diagram equivalence

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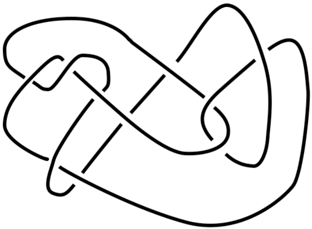

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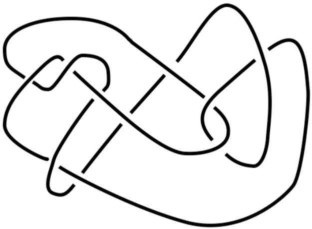

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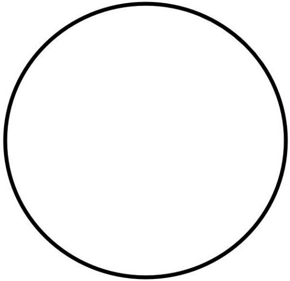
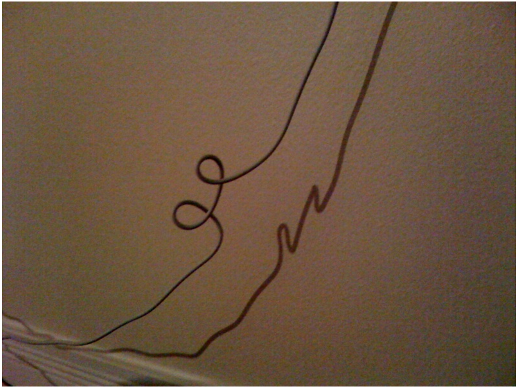


...is the unknot in disguise!

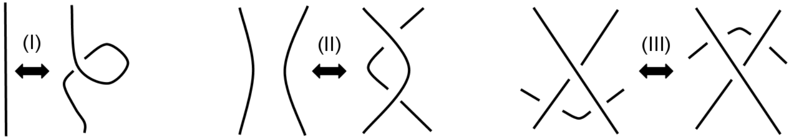
Knot diagram equivalence

How can we show that two diagrams represent the same knot?



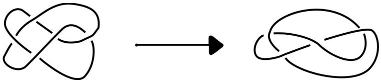
Reidemeister moves

In 1927, Kurt Reidemeister showed that knot diagrams are equivalent precisely when they can be related by the following moves.



I'll make an example of you!

Let's use the example we looked at before to show how Reidemeister moves work.



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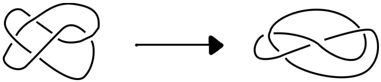

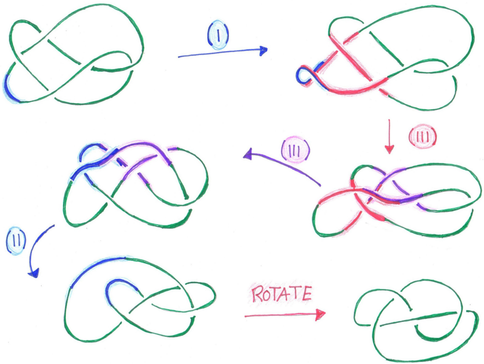

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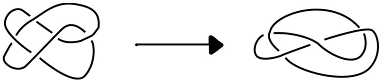

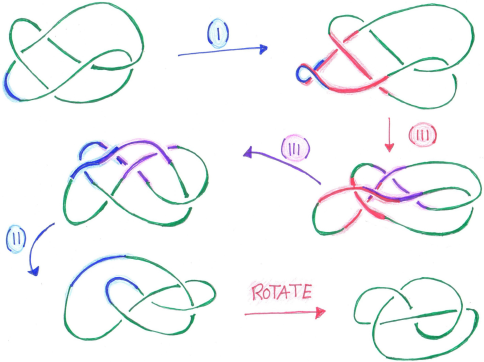

At the end of our sequence of moves, we have the **mirror image** of the diagram we wanted.

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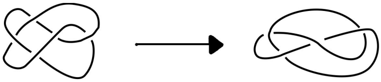
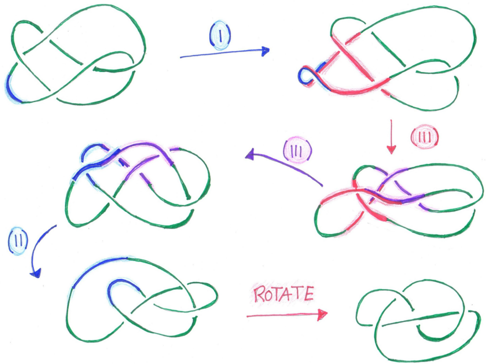
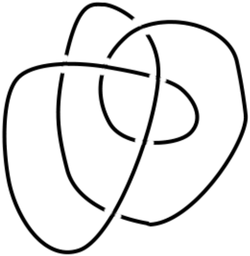


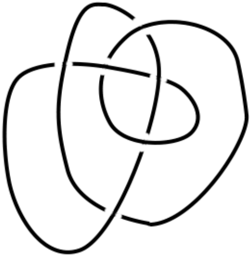


At the end of our sequence of moves, we have the **mirror image** of the diagram we wanted. This knot, called the **figure eight knot**, is equivalent to it's mirror image. (Prove it!)

A slippery slope

Just as the same knot can look very different in two different diagrams, different knots can look very similar to one another.





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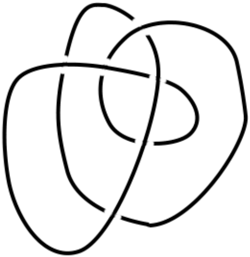
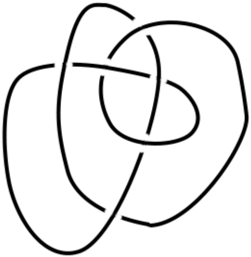


Unknotted



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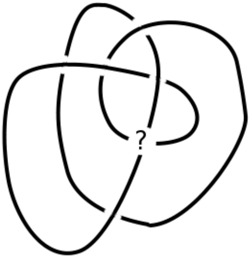

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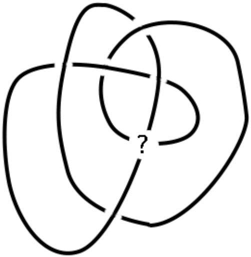

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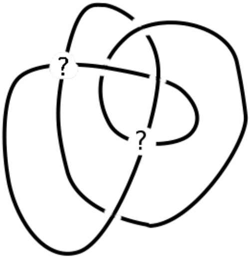

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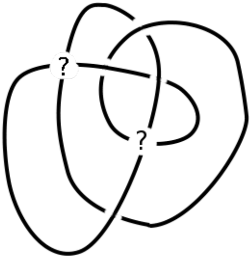

Knotted??

Let's use this idea to play a game!

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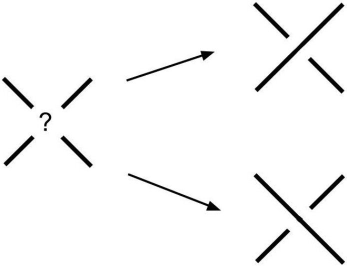

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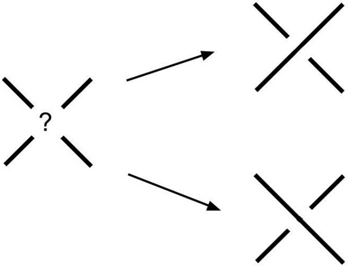
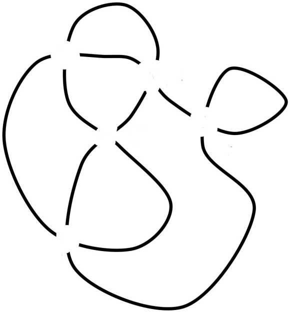


- KNOT wants to make something that is knotted up, while UNKNOT wants to make something that can be untangled.

Playing the game



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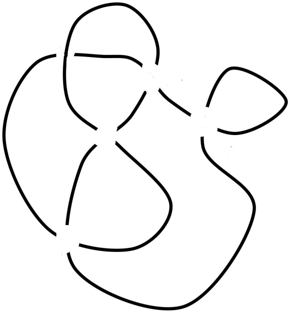
Unknot's Move



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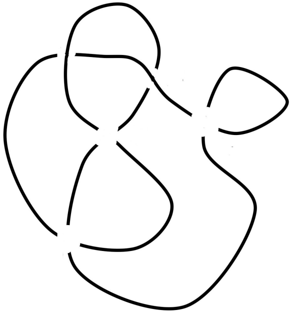

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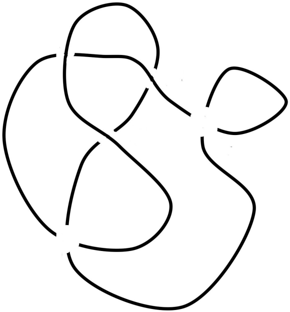

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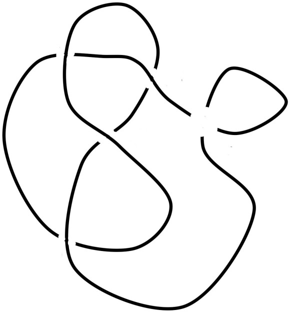

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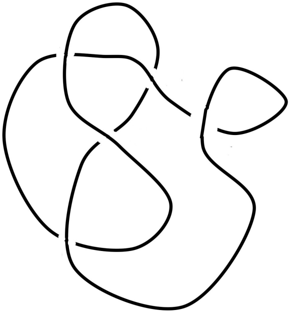

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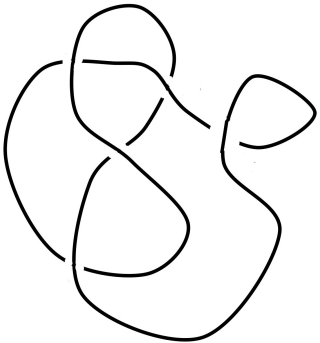

WHO WINS?

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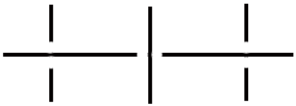
8. Did any questions arise?

Some observations

- If *Knot* can make the knot *alternating*, she can win.



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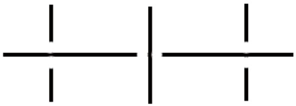
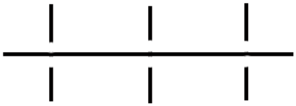


- If *Unknot* can make long strands go entirely under or entirely over, the knot can be simplified.



A winning strategy

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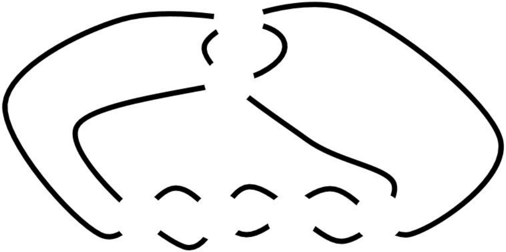

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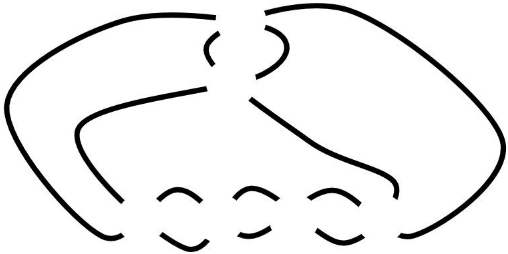

- What about your game board? Who has a winning strategy?

What other games could we play?

Knotting vs. Linking

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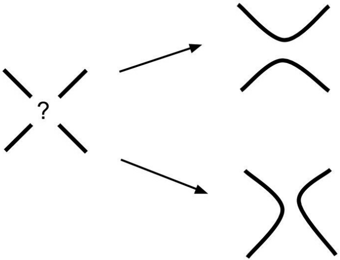

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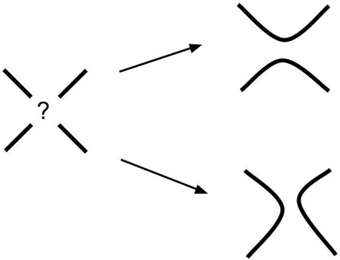
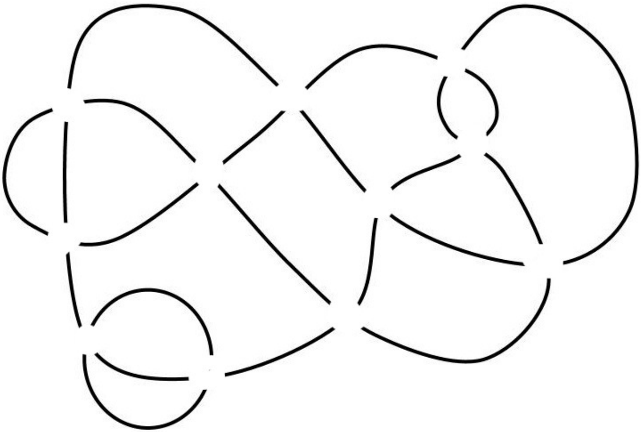
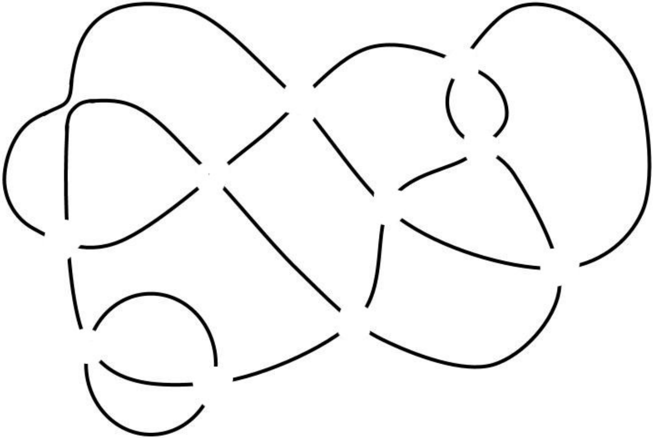


- *Link* wants to disconnect the the diagram to get an unlink, while *Knot* wants to keep it all in one piece to get an unknot.

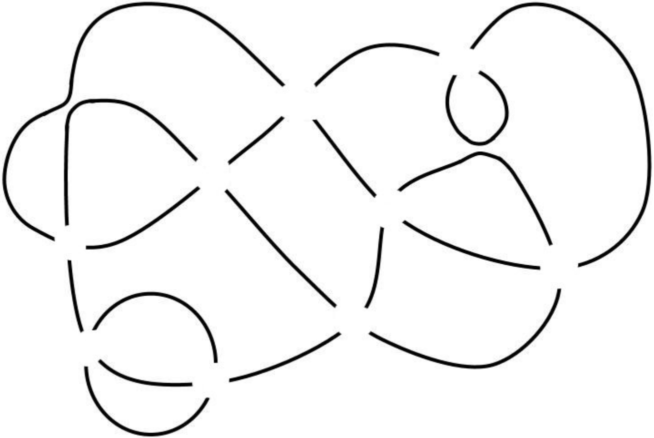
Playing the game



Knot's Move



Link's Move



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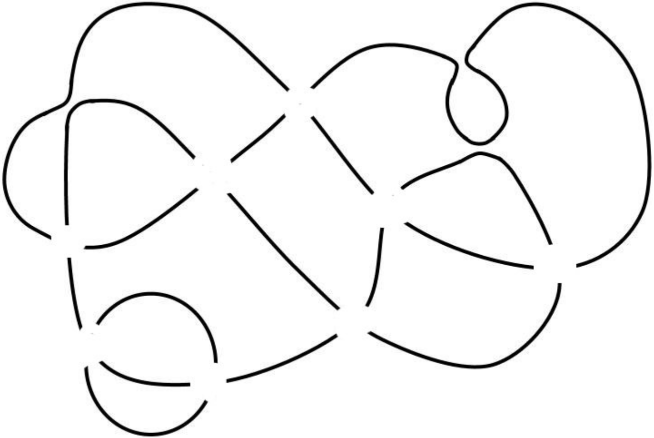
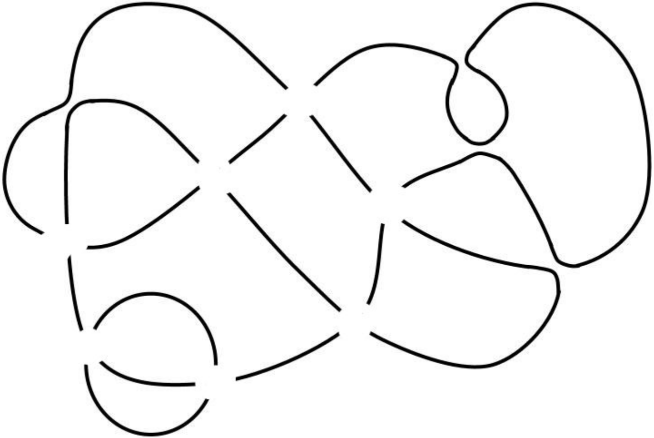


Link's Move



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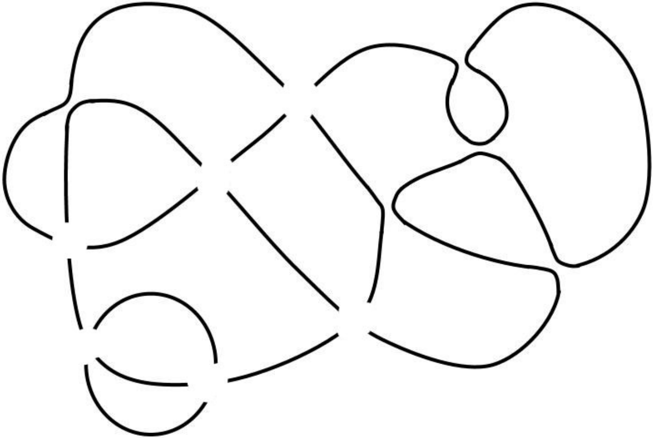

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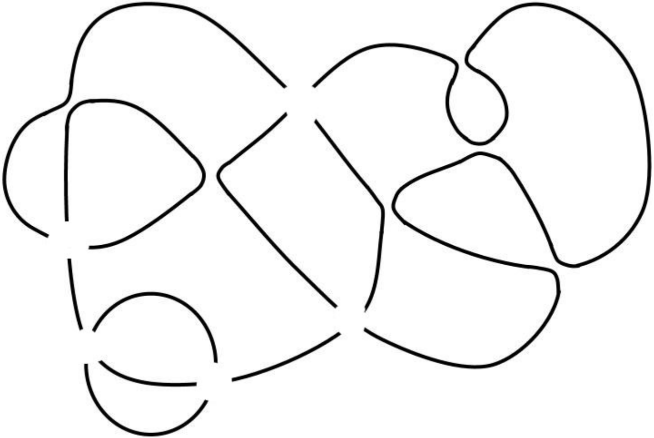

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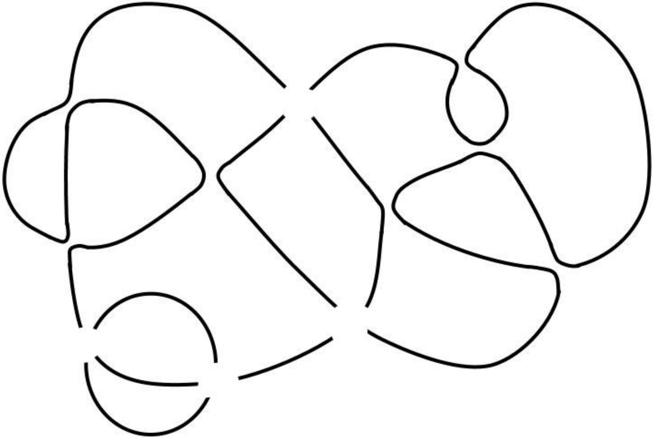

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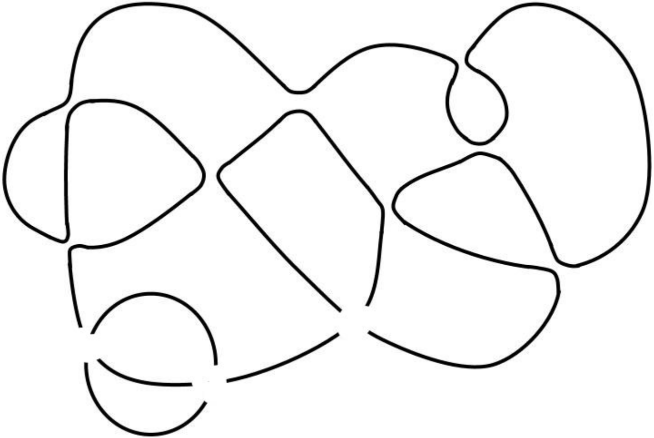

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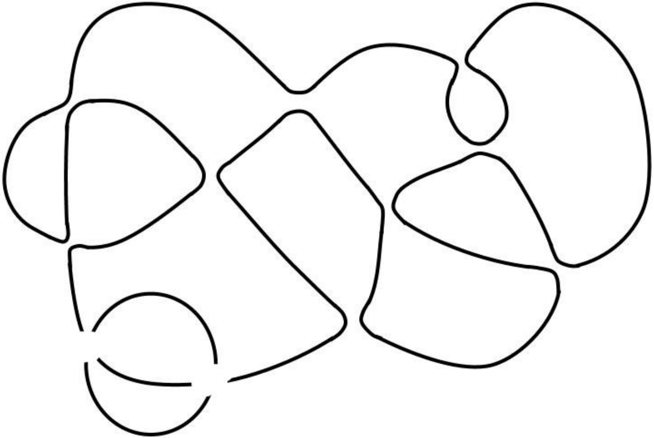

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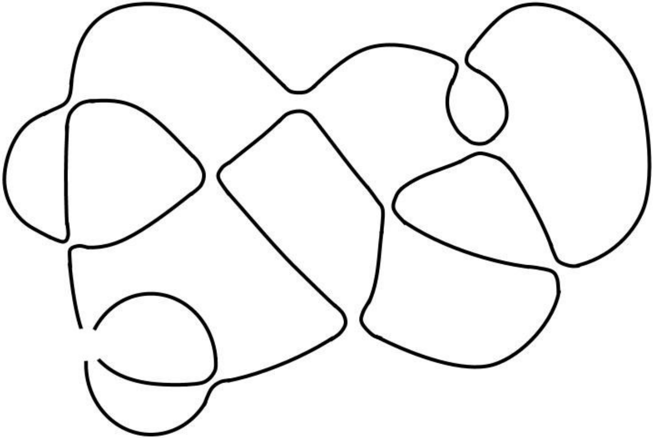

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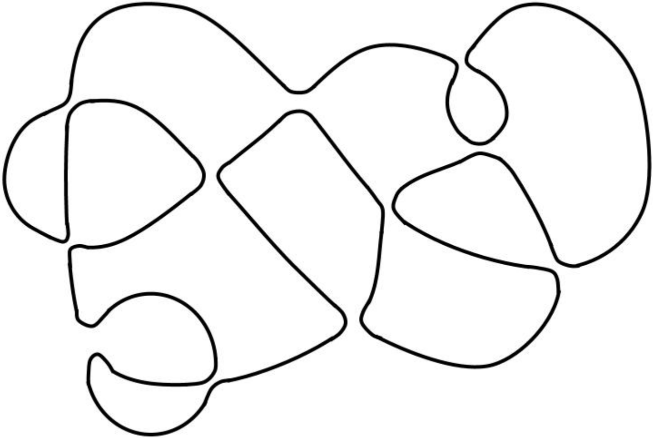

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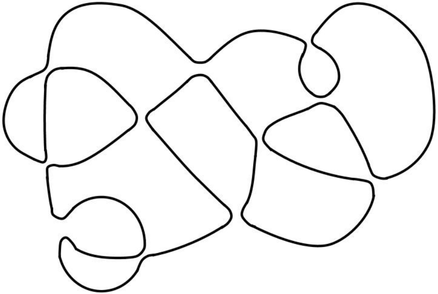

WHO WINS?

Now it's your turn

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   (One of you is *Knot* and one of you is *Link*.)

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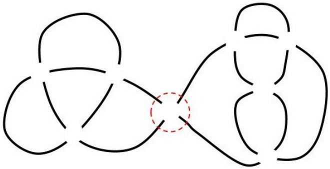
6. Did any questions arise?

Some observations

- *Link* wins if he can play on a **nugatory** crossing.



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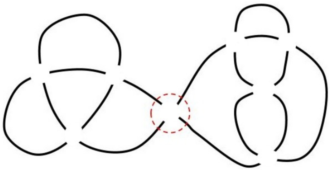

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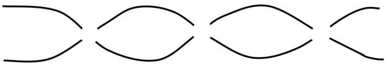

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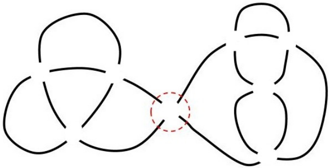

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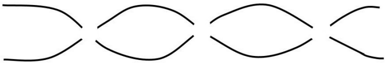

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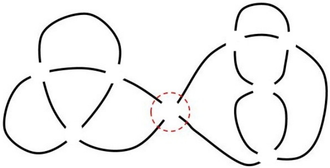

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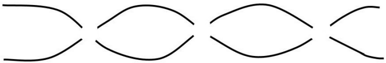
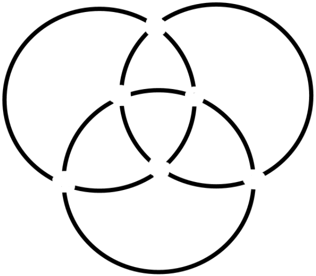


- When *Link* plays last, *Link* wins.
- Does *Link* always have the upper hand??

A winning strategy

- This is an example of a link shadow where *Knot* actually has a winning strategy if she plays second.



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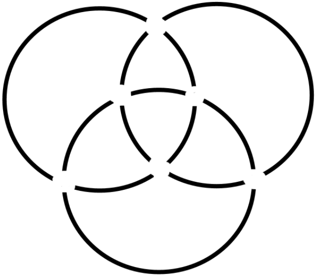

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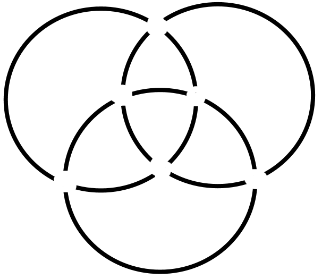

- More often than not, *Link* has a winning strategy...but we have found infinite families of diagrams on which *Knot* has an advantage.

Unsolicited advice

- **Keep playing these games** and see if you can figure out who has a winning strategy for specific shadows!

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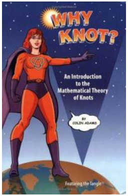
- These are just a couple of games you could play using knots and links. **Invent your own games**!
- Are you interested in knowing more about knots? **Read** *Why Knot?*, a comic book about knots by Colin Adams.



thank you.
thank you.
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