You're my better half

a tale of ~~complimentary~~ complementary sequences

Tom Edgar, May 17 2020

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$(1,1,2,3,5,8,13\ldots) = (F_n)_{n=1}^{\infty}$ (Fibonacci numbers)

$$F_n = F_{n-1} + F_{n-2}$$

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For instance, we don't know if $e + \pi$ is irrational.

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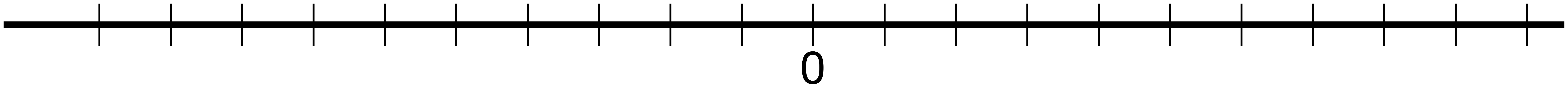

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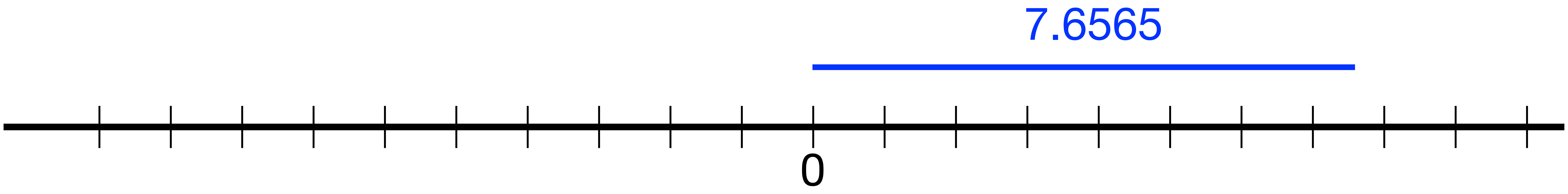

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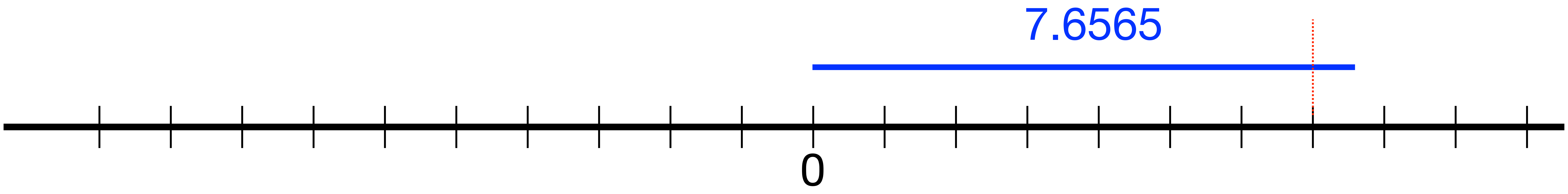

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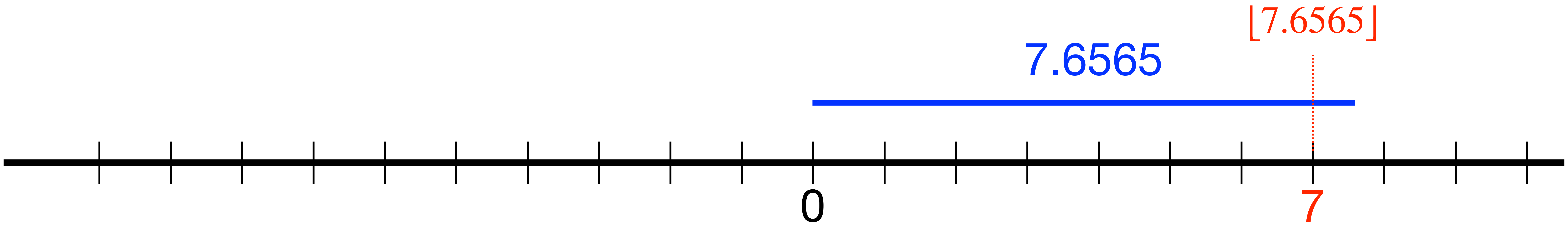

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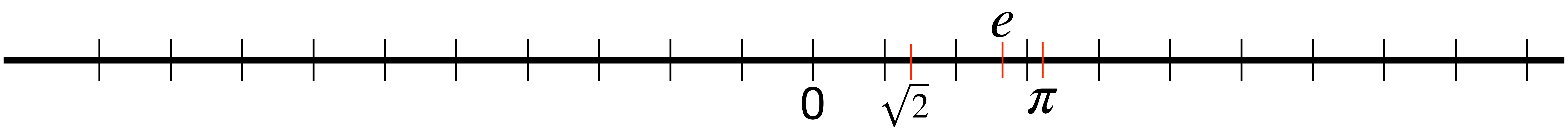
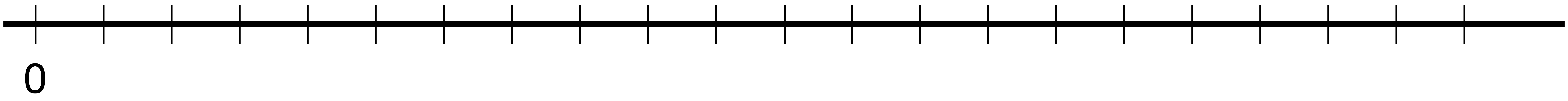


$$\lfloor \sqrt{2} \rfloor = 1 \qquad \lfloor e \rfloor = 2 \qquad \lfloor \pi \rfloor = 3$$

Consider multiples of $\sqrt{2}$



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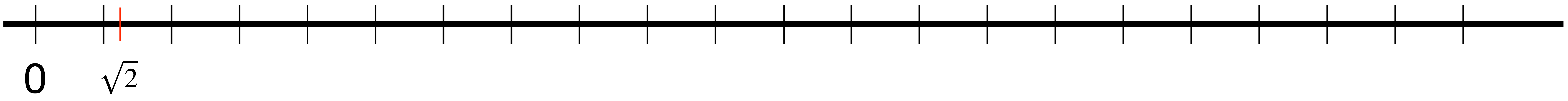

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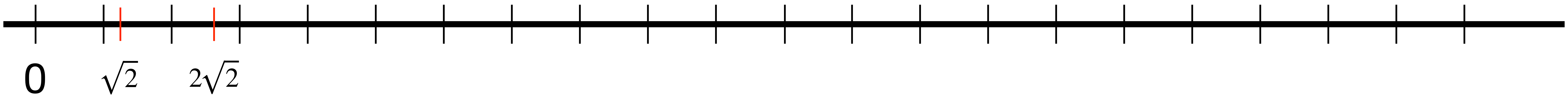

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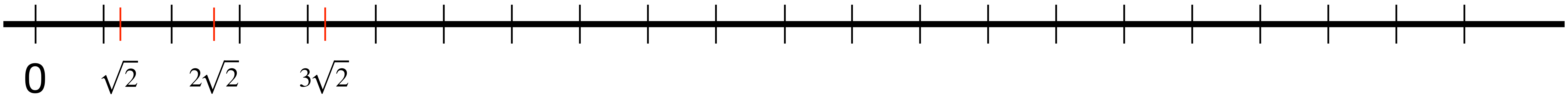

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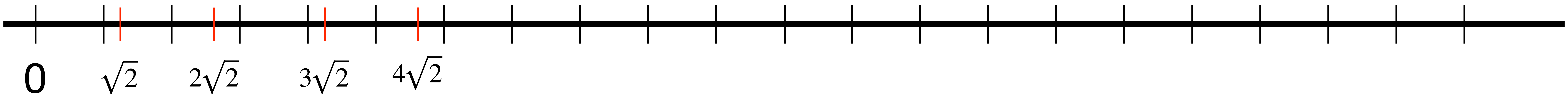

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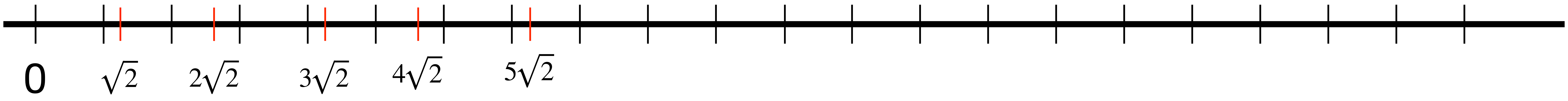

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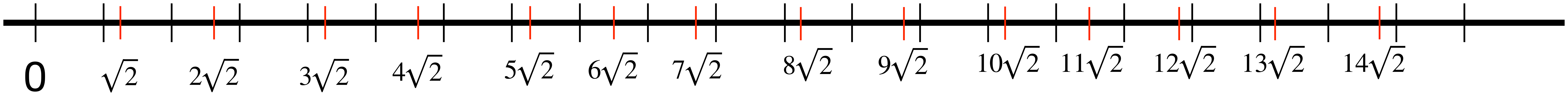

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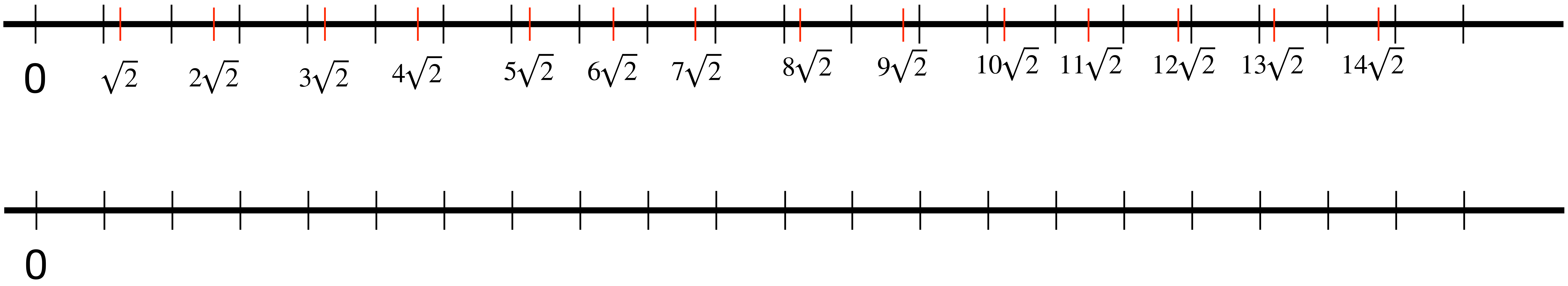

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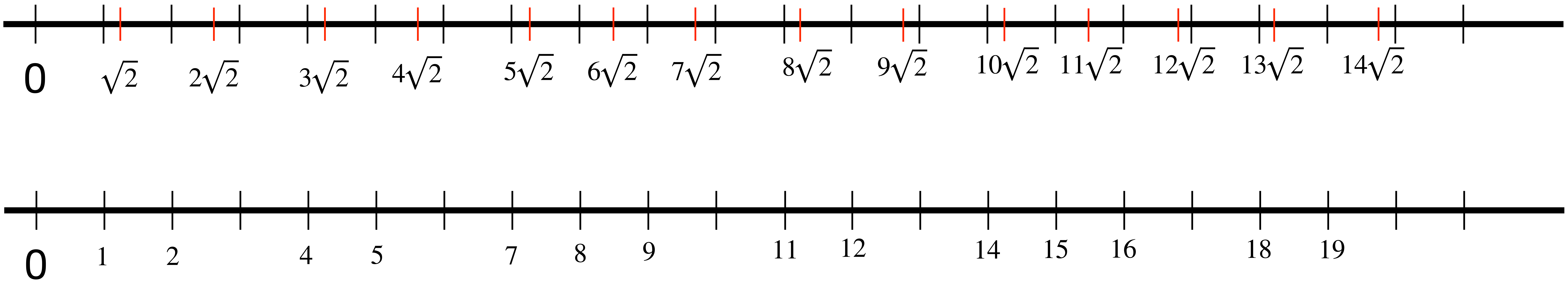

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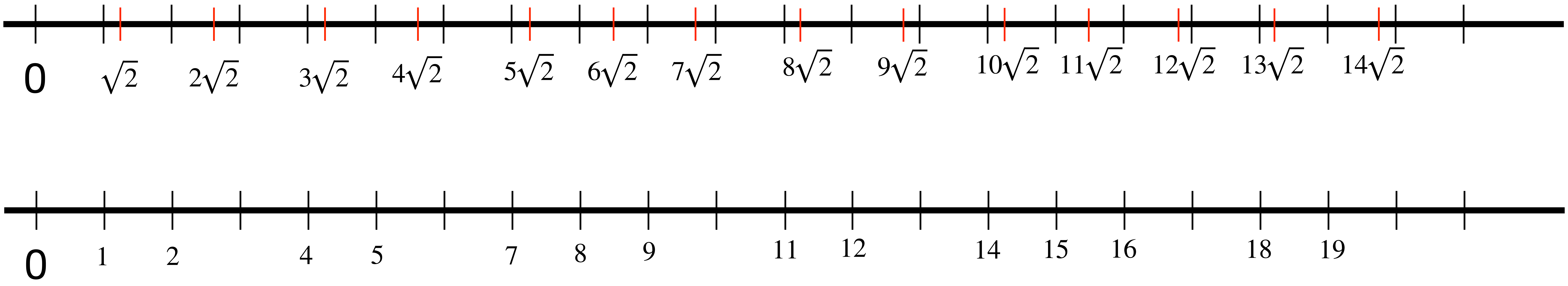

$$\left(\lfloor n \cdot \sqrt{2} \rfloor\right)_{n=1}^{\infty} = (1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, \ldots)$$

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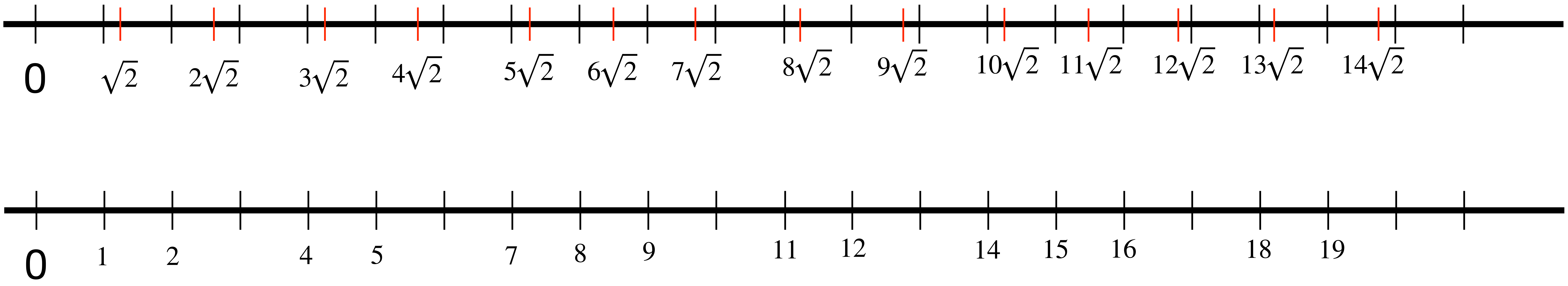

$$\left(\lfloor n \cdot \sqrt{2} \rfloor\right)_{n=1}^{\infty} = (1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, \ldots)$$

A mystery: what numbers are skipped?

Finding the missing numbers

$$\left(\lfloor n \cdot \sqrt{2} \rfloor\right)_{n=1}^{\infty} = (1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, ...)$$

Construct the analogous sequence of multiples of $\dfrac{\sqrt{2}}{\sqrt{2}-1}$

$$\left(\left\lfloor n \cdot \frac{\sqrt{2}}{\sqrt{2}-1} \right\rfloor\right)_{n=1}^{\infty} = (3, 6, 10, 13, 17, 20, ...)$$

Finding the missing numbers

(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, ...)

Every positive integer appears once and only once!

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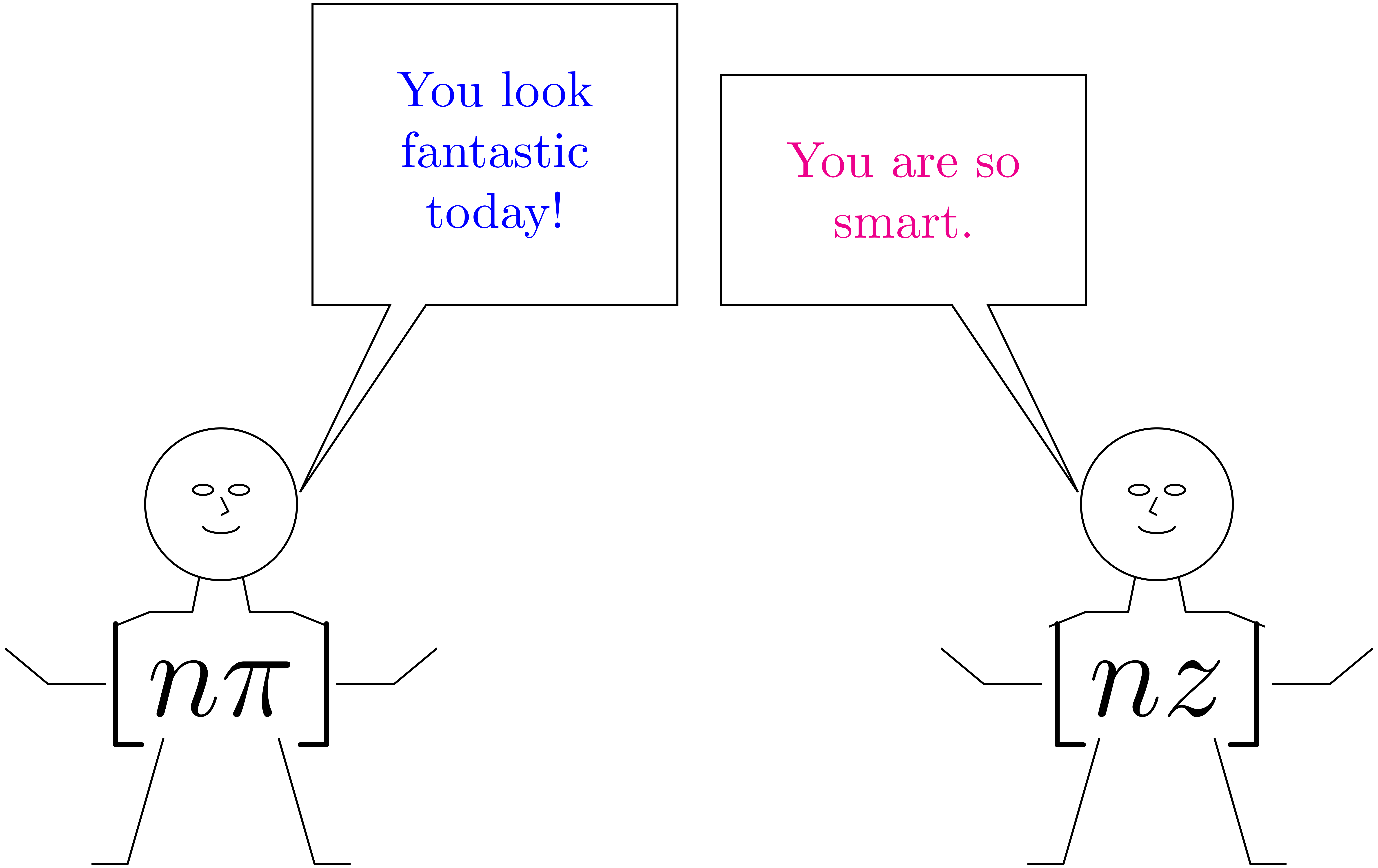

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$$\frac{1}{\sqrt{2}} + \frac{1}{\frac{\sqrt{2}}{\sqrt{2}-1}} = \frac{1}{\sqrt{2}} + \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{1+\sqrt{2}-1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

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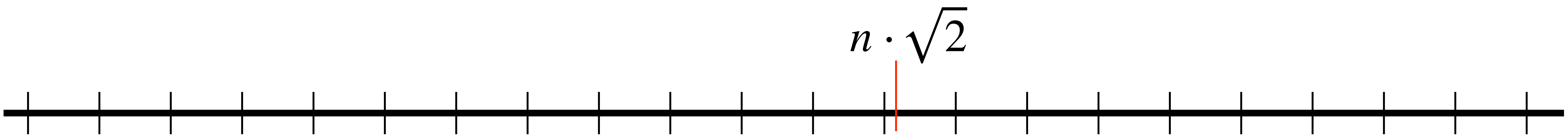

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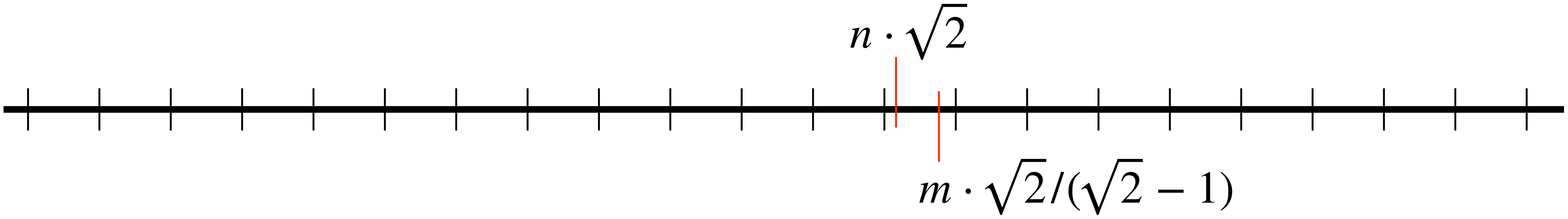

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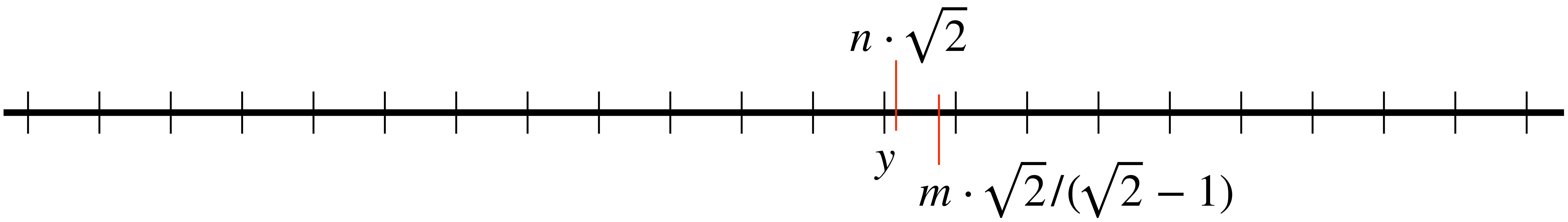

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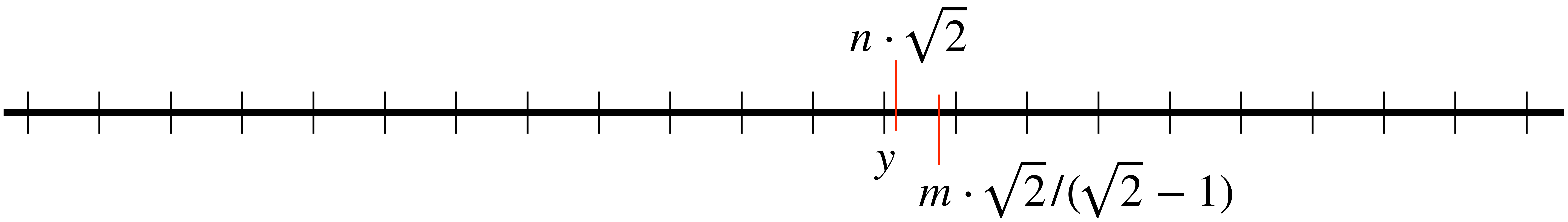

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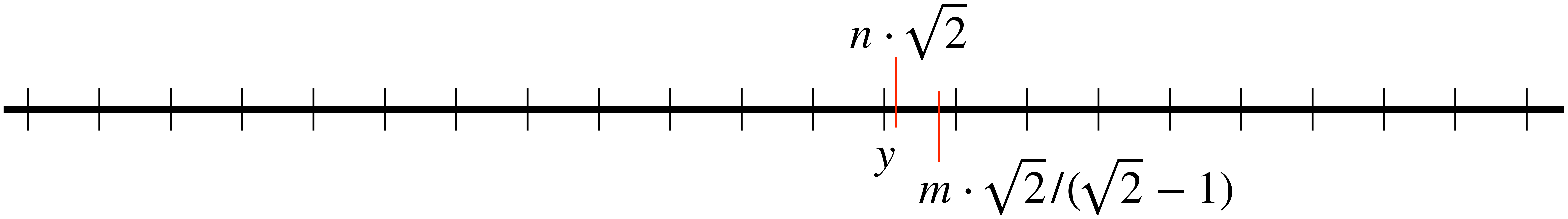

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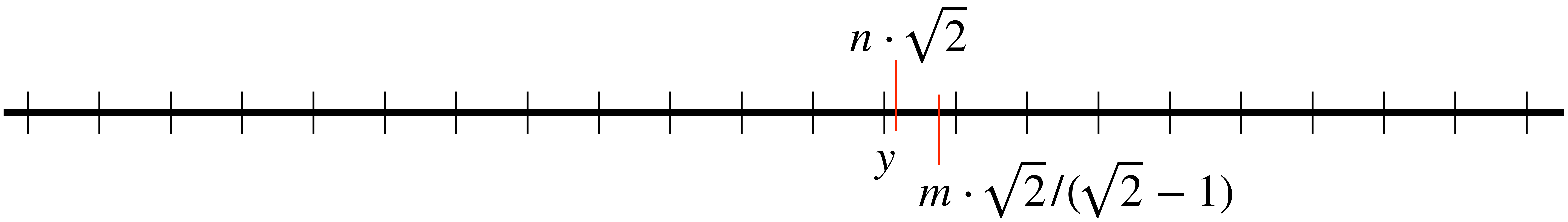

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- $\sqrt{2}$ and $\sqrt{2}/(\sqrt{2} - 1)$ are both irrational;
- any nonzero integer multiple of an irrational is also irrational.

No Collisions

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$$\frac{y}{\sqrt{2}} < n < \frac{y + 1}{\sqrt{2}}$$

No Collisions

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counting numbers

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Suppose there exists a counting number y not in either sequence.

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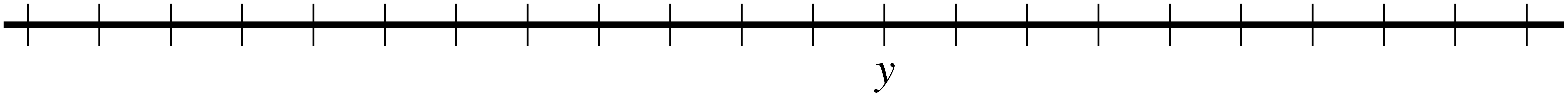

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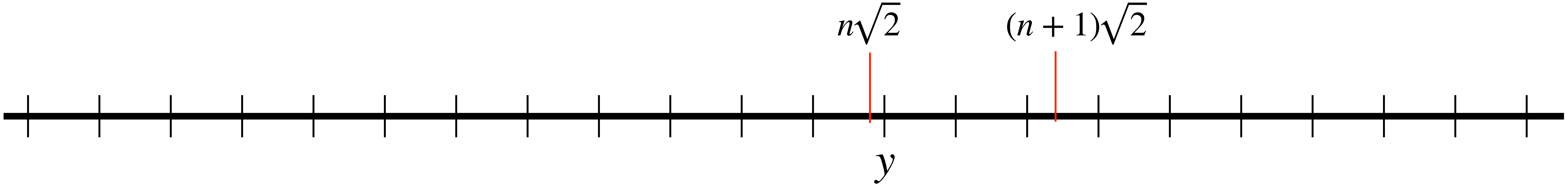

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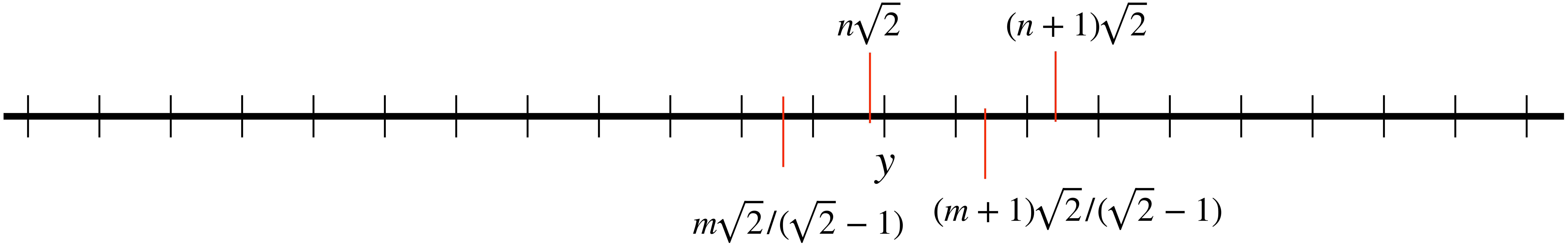

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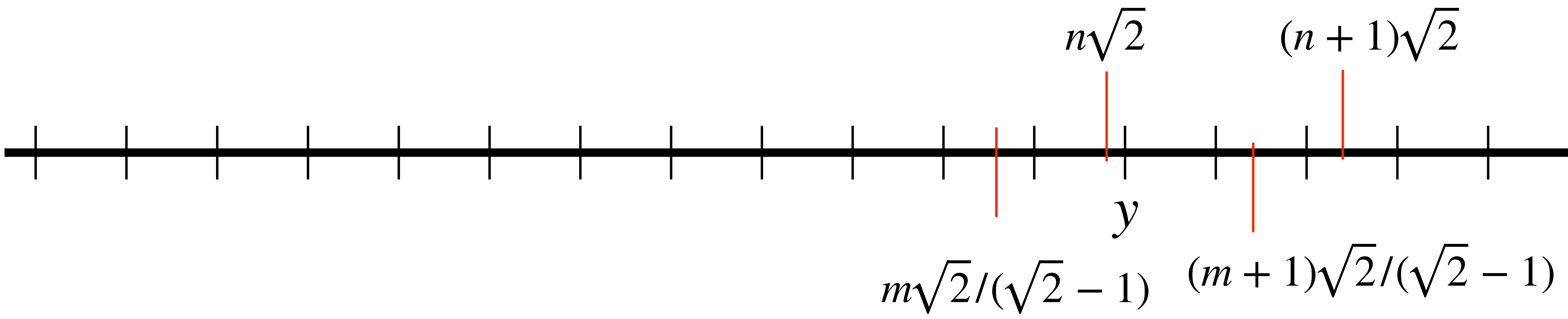

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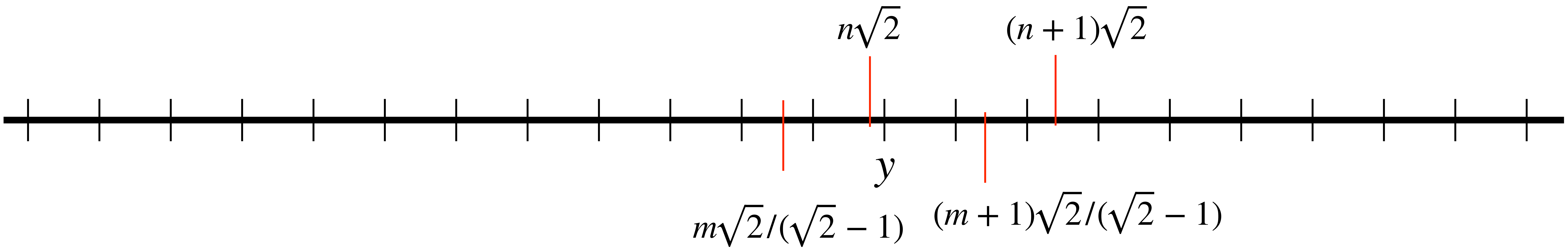

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Given a sequence (like the squares), we get the complement by taking everything else:

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Question: Are there other methods to generate complementary sequences?

Yes! Let's discuss a general method

How to find complementary sequences

Suppose that $f(n)$ is an increasing integer sequence, such as $f(n) = 2n$ plotted below.

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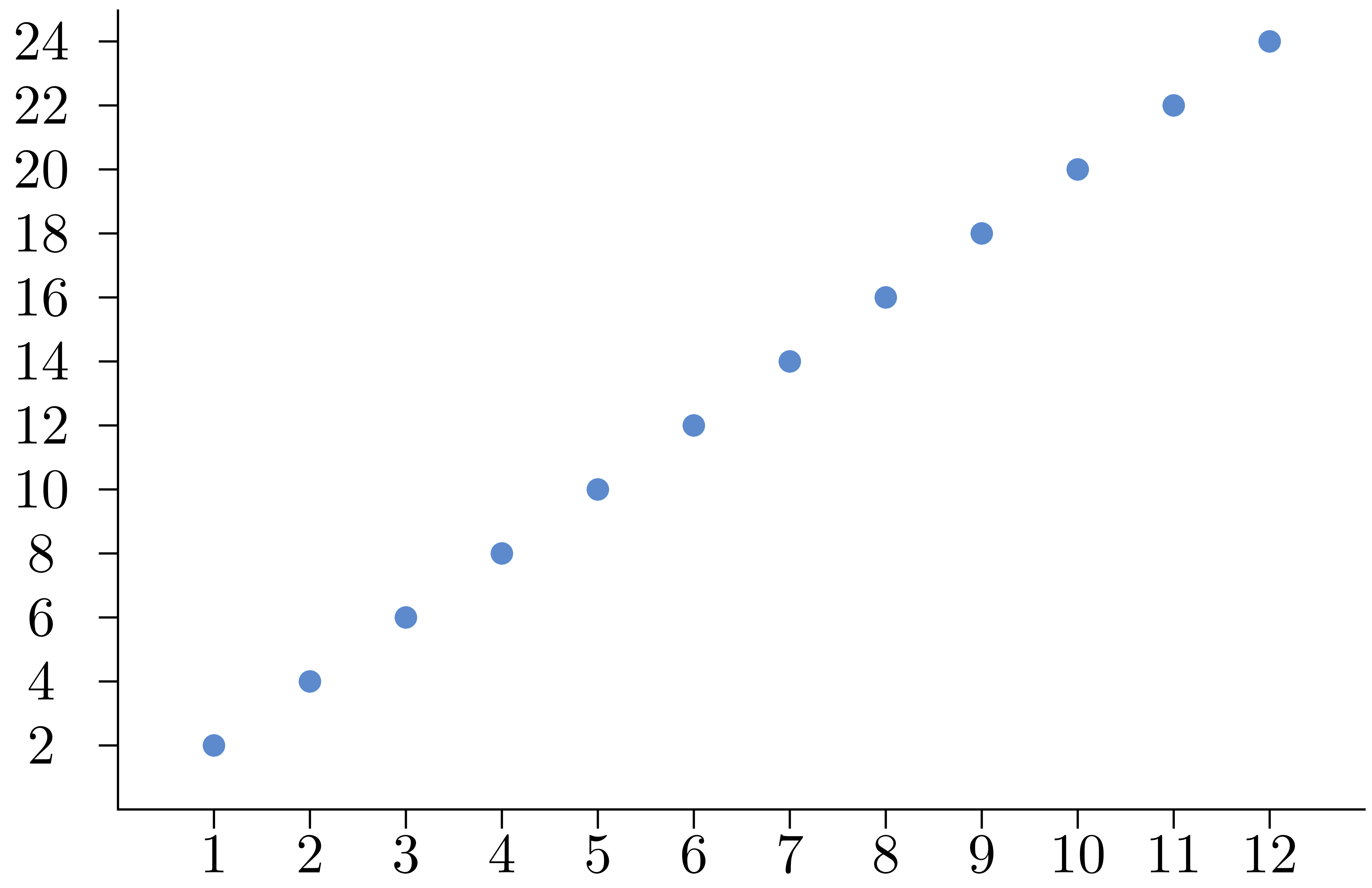

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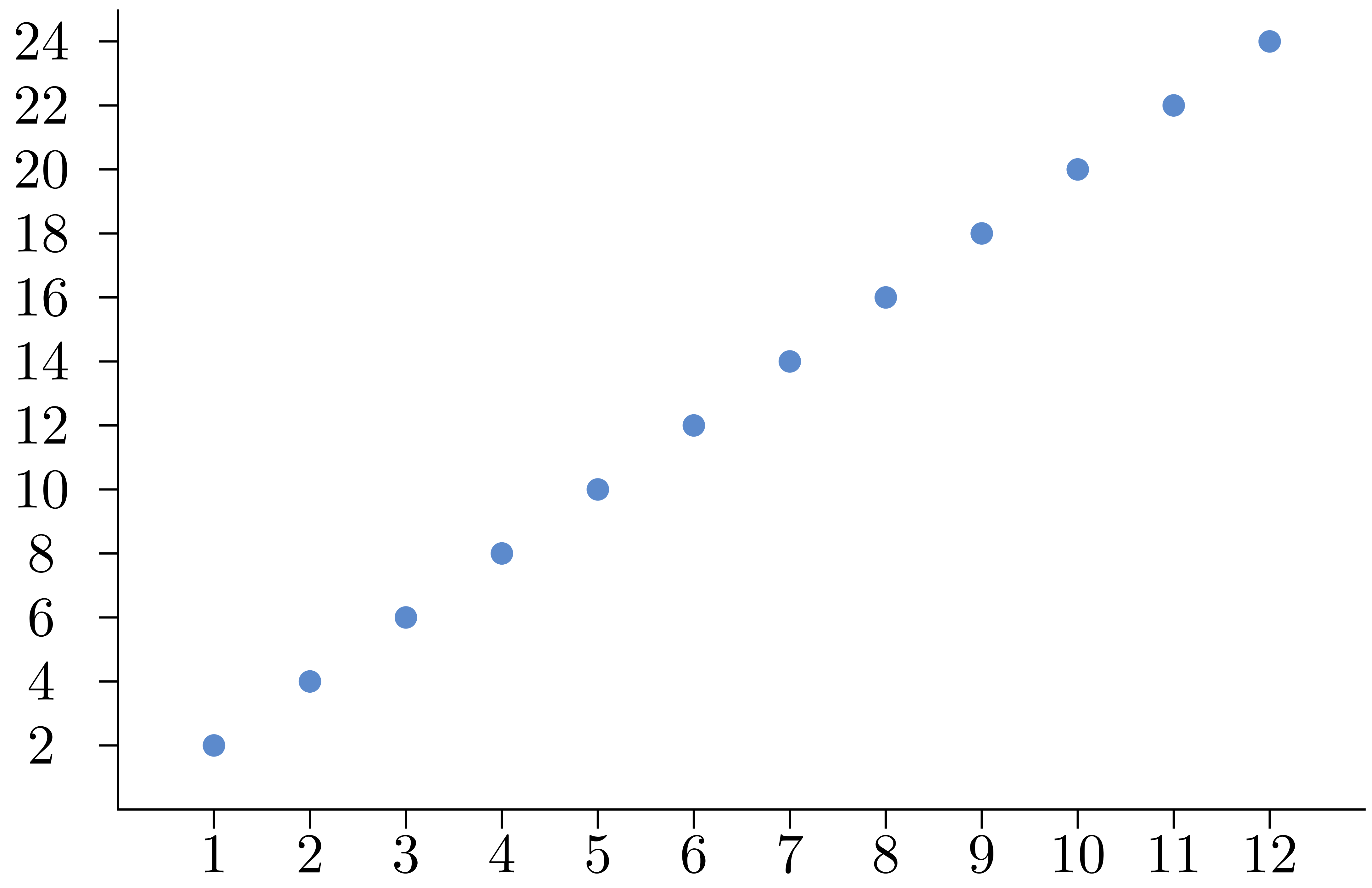

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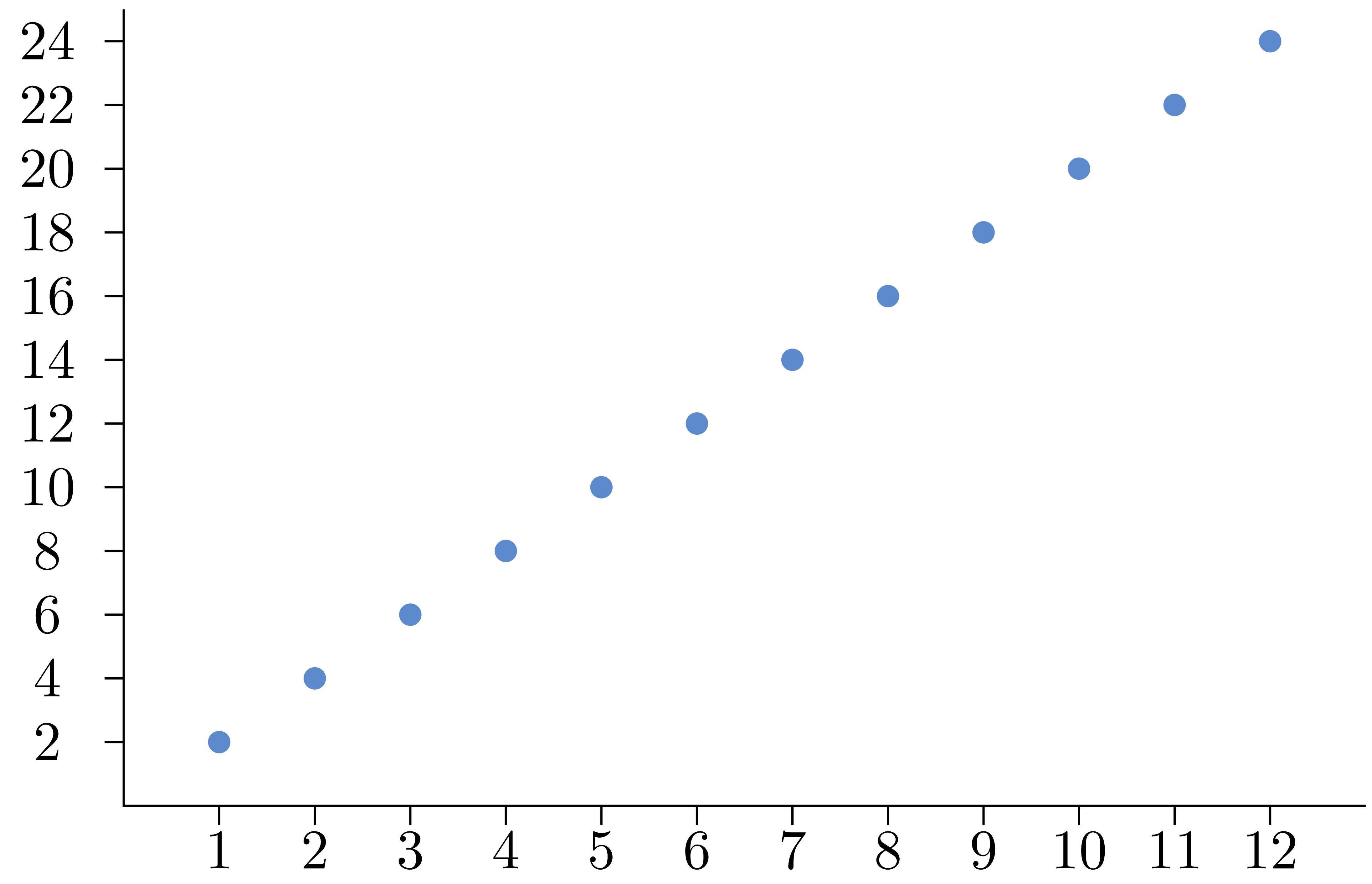

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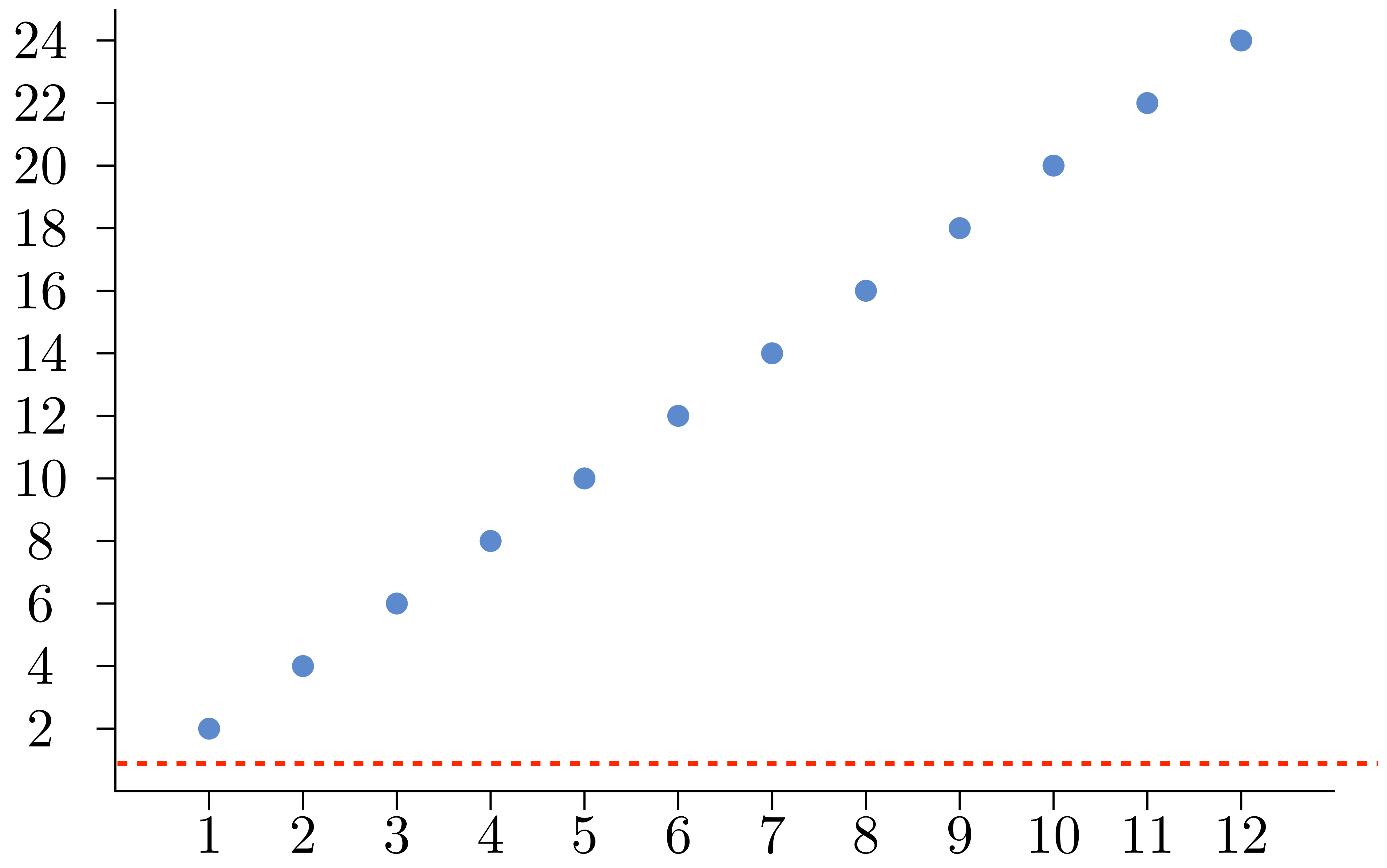

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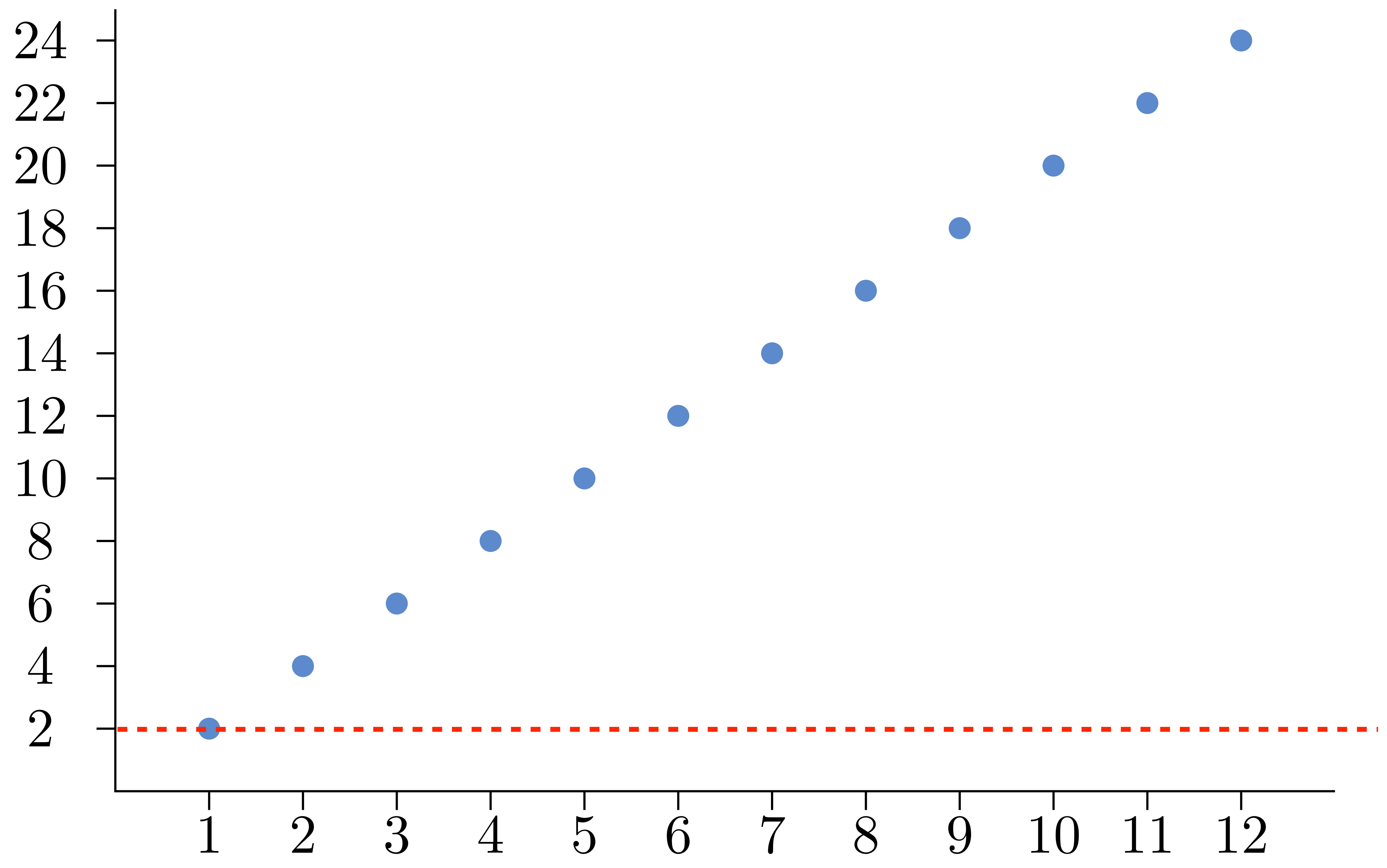

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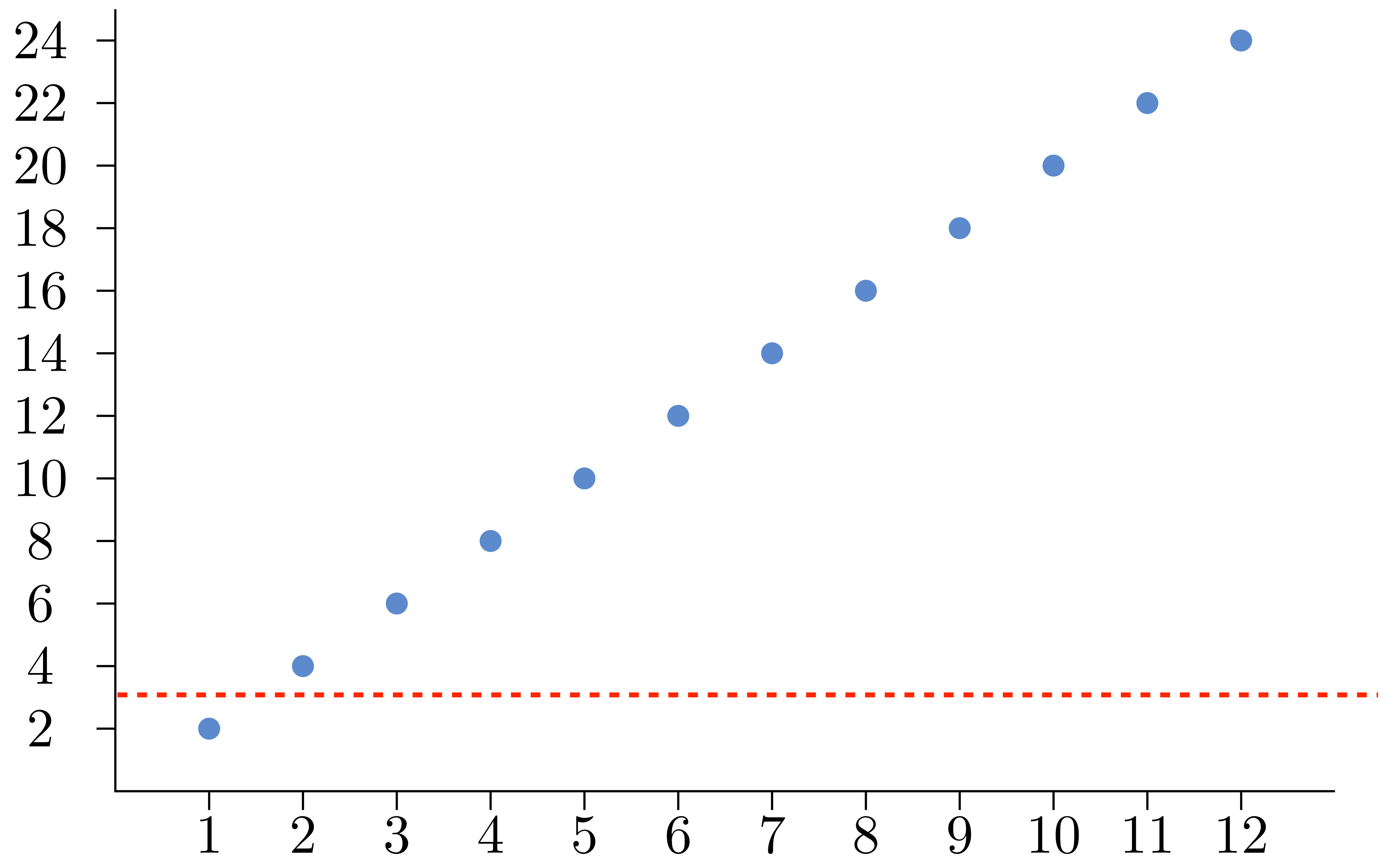

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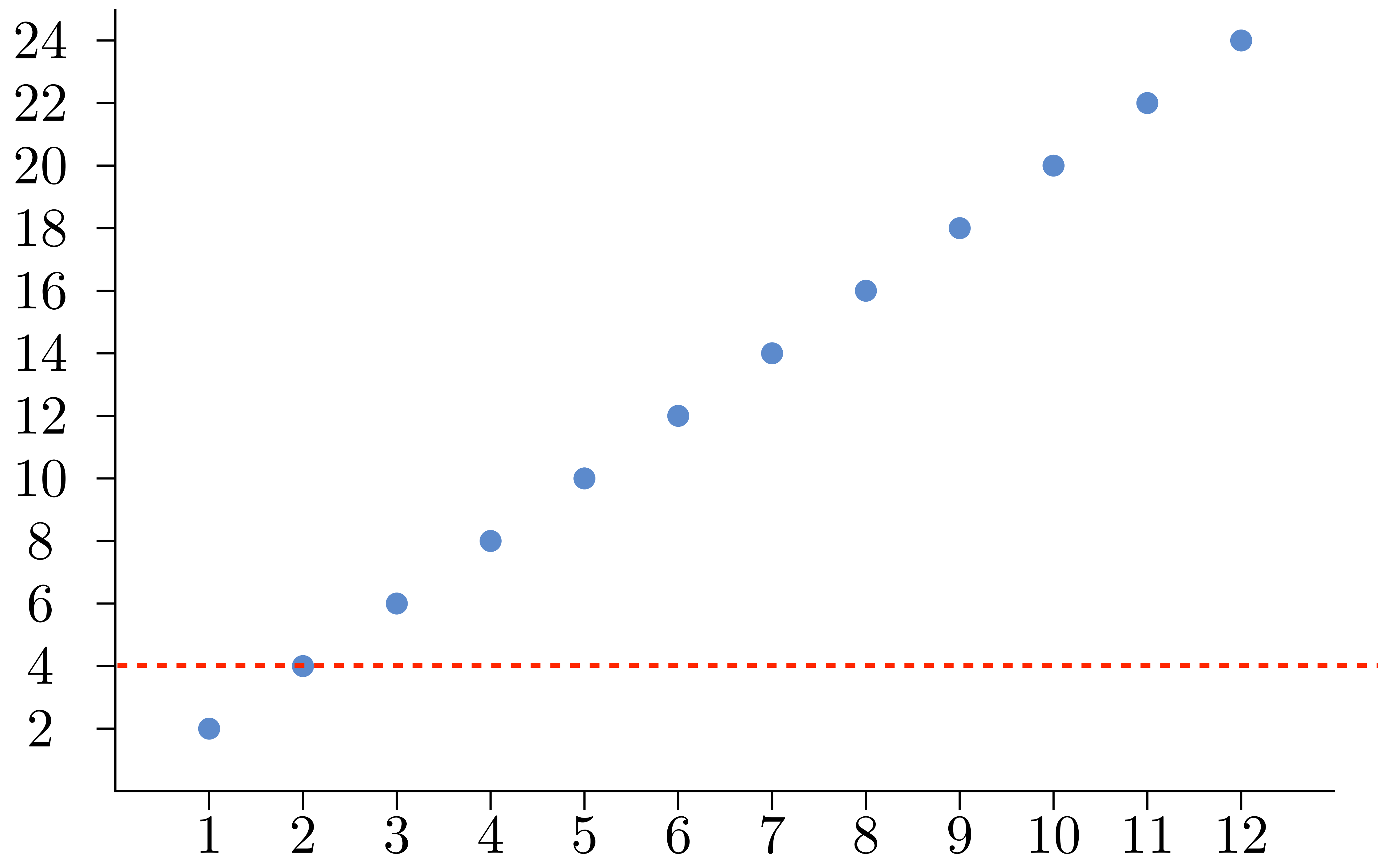

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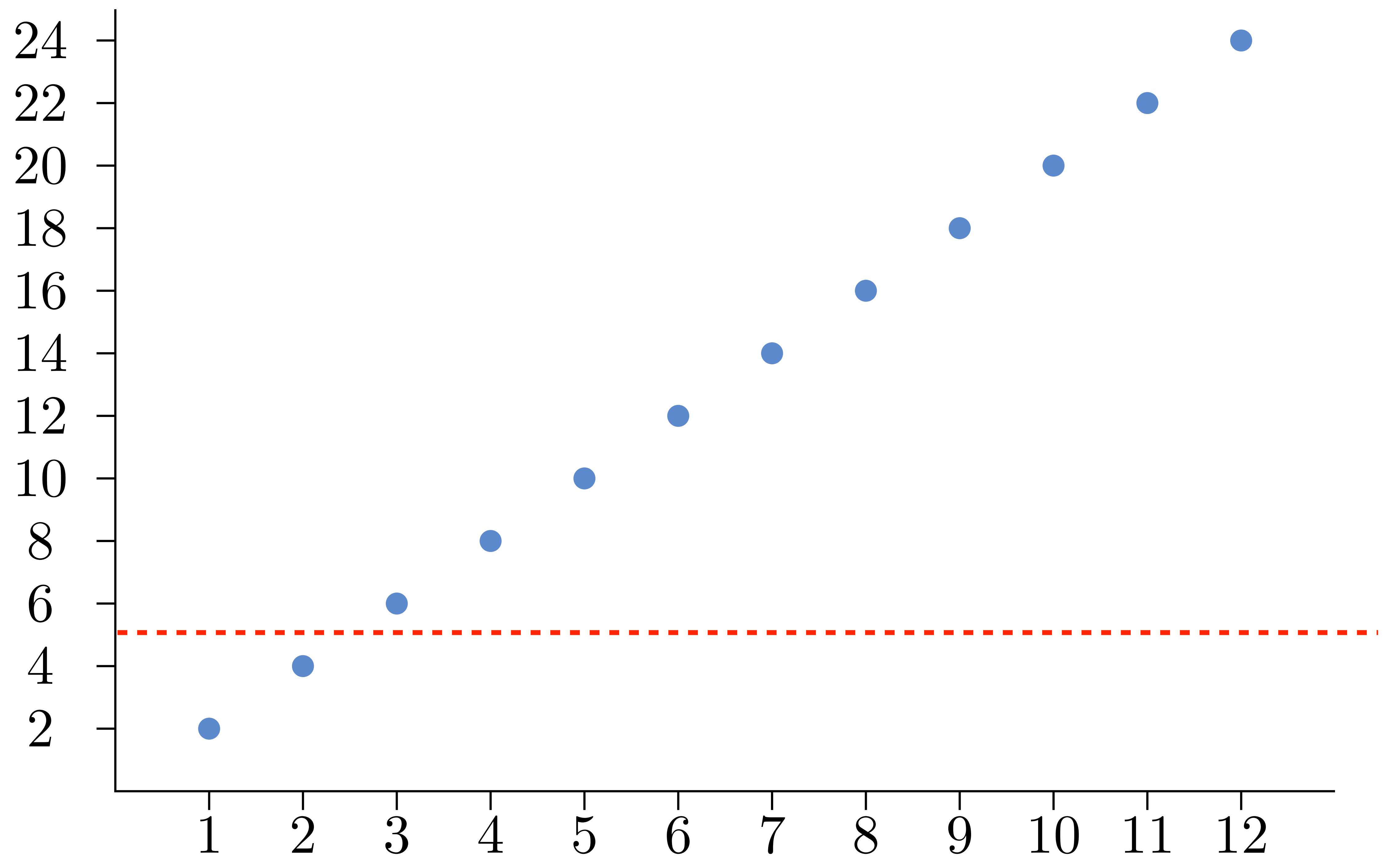

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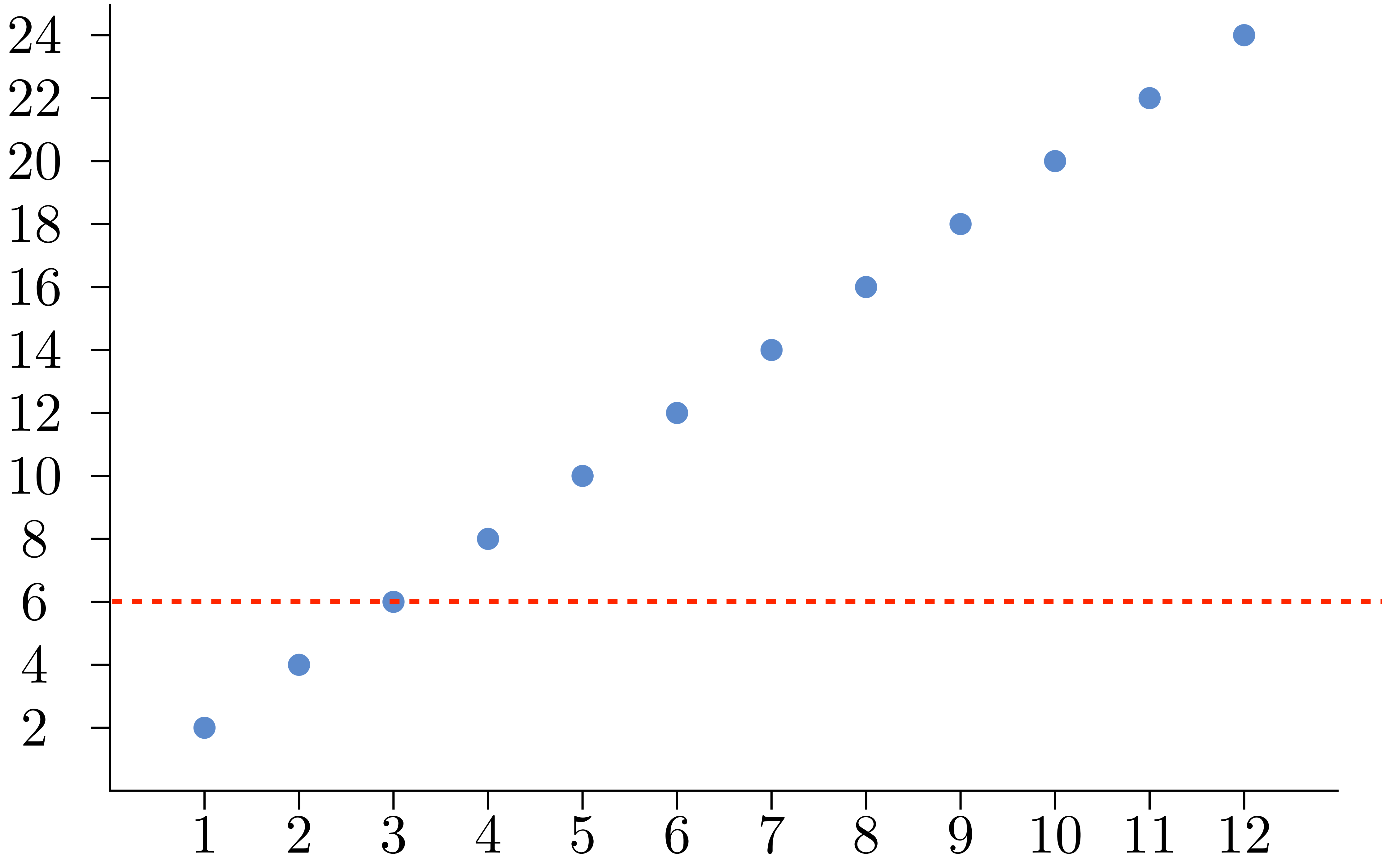

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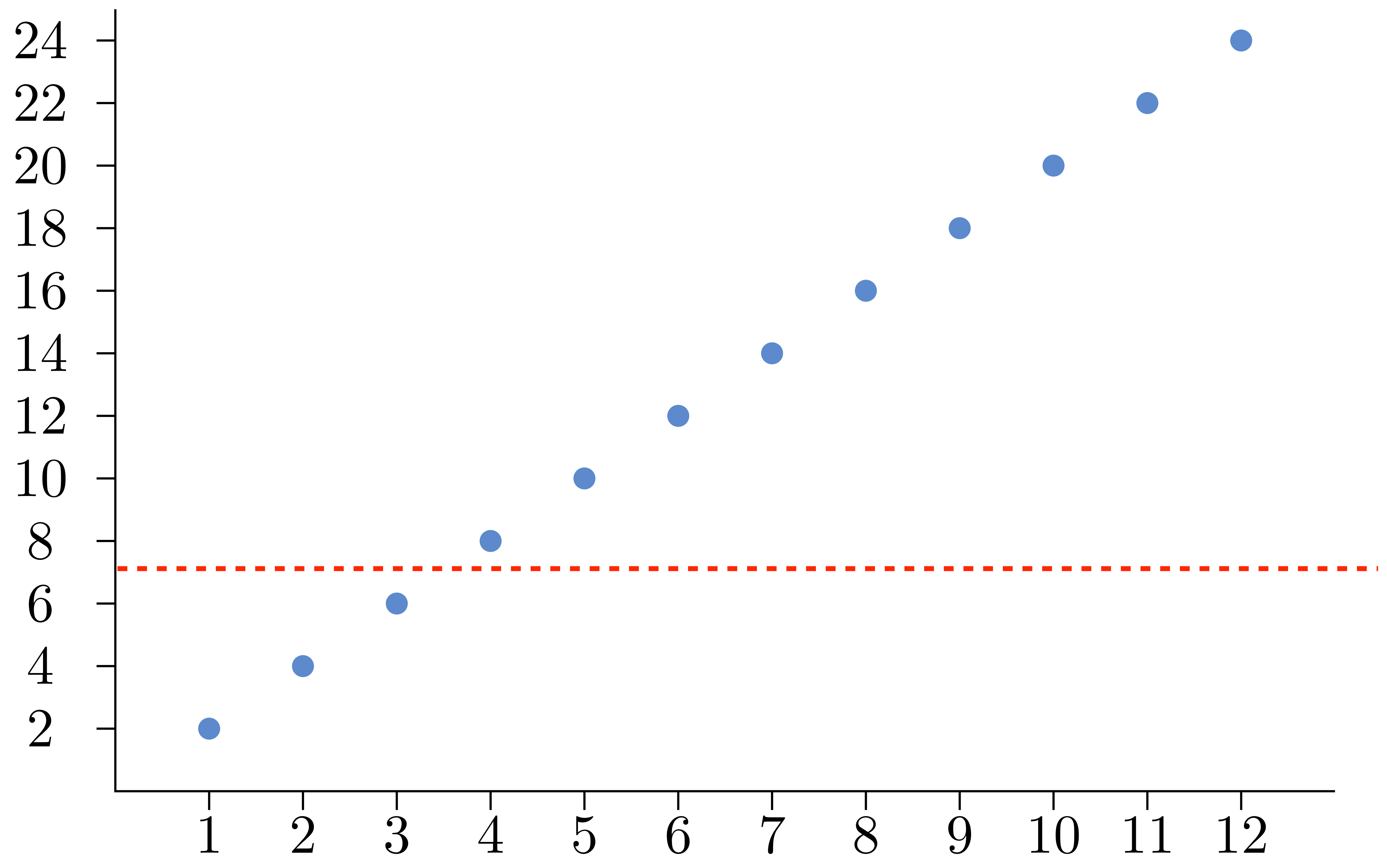

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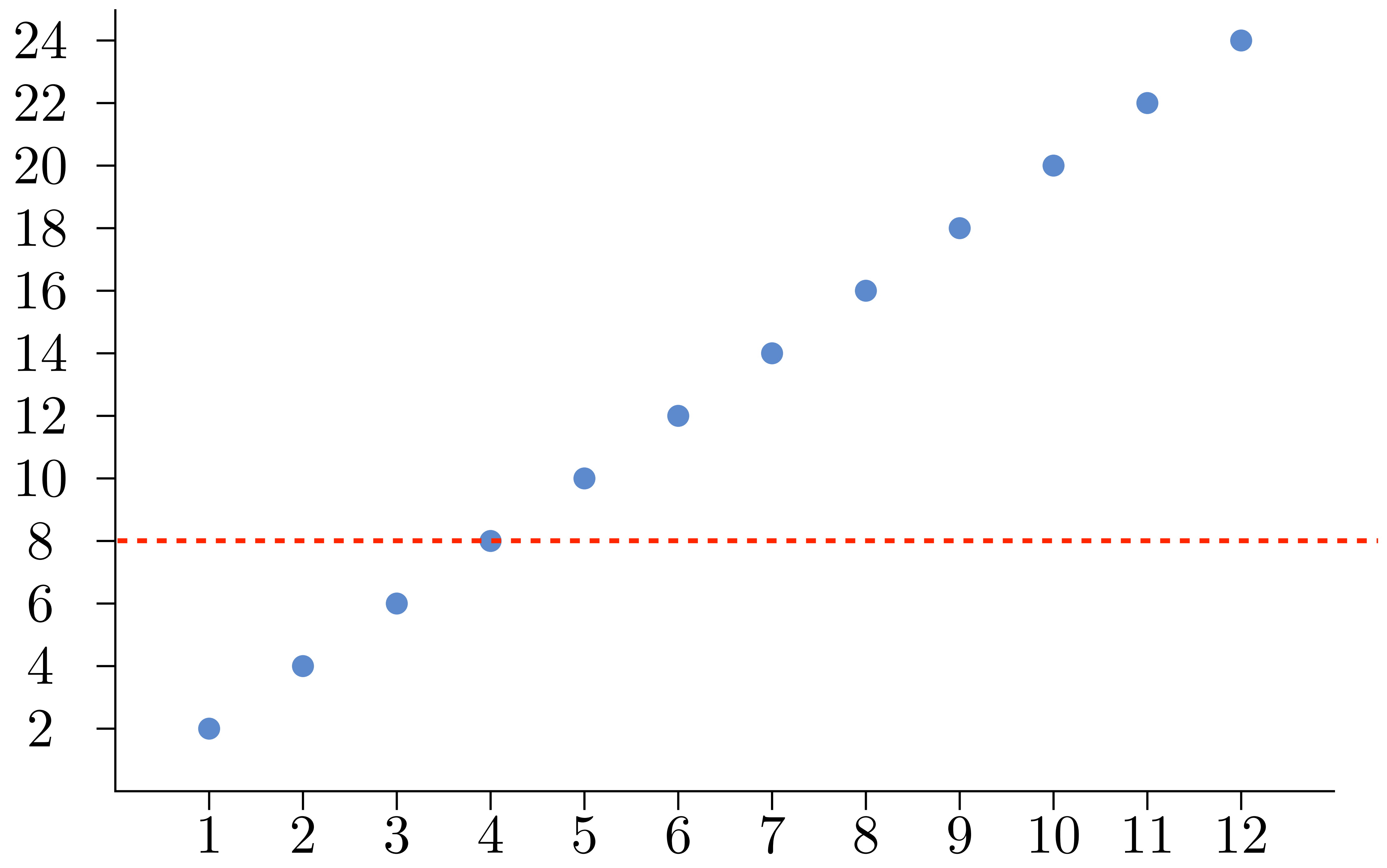

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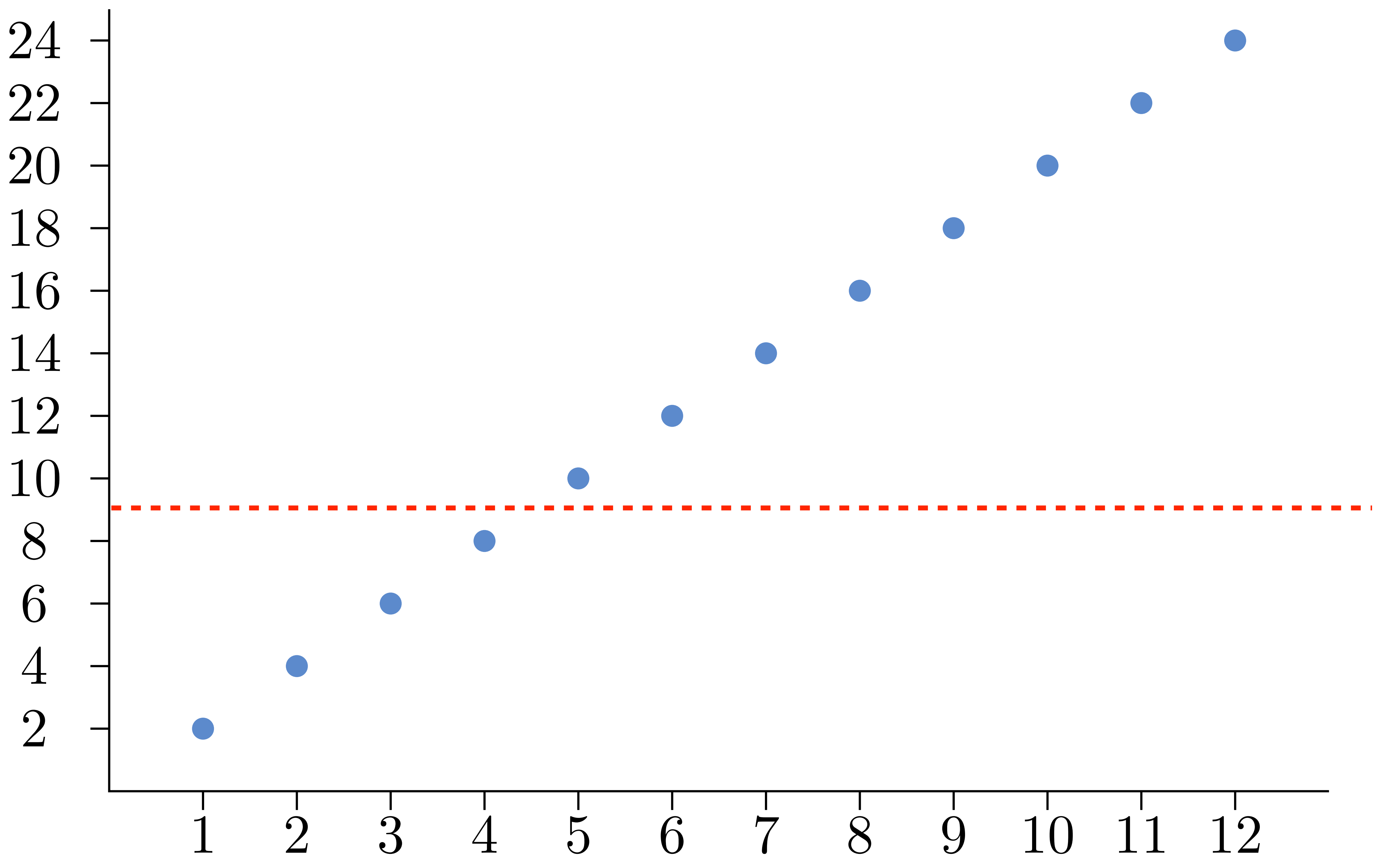

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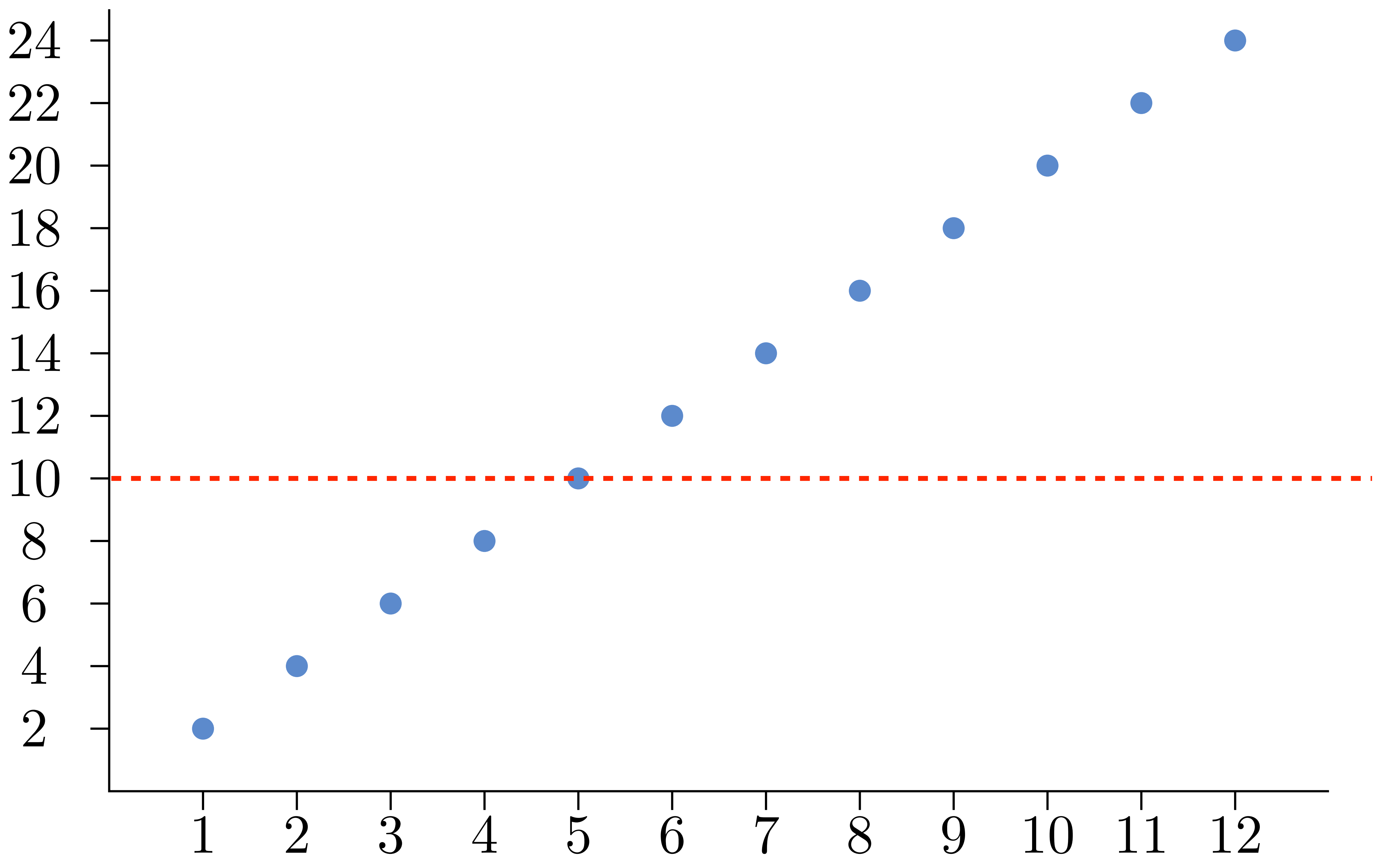

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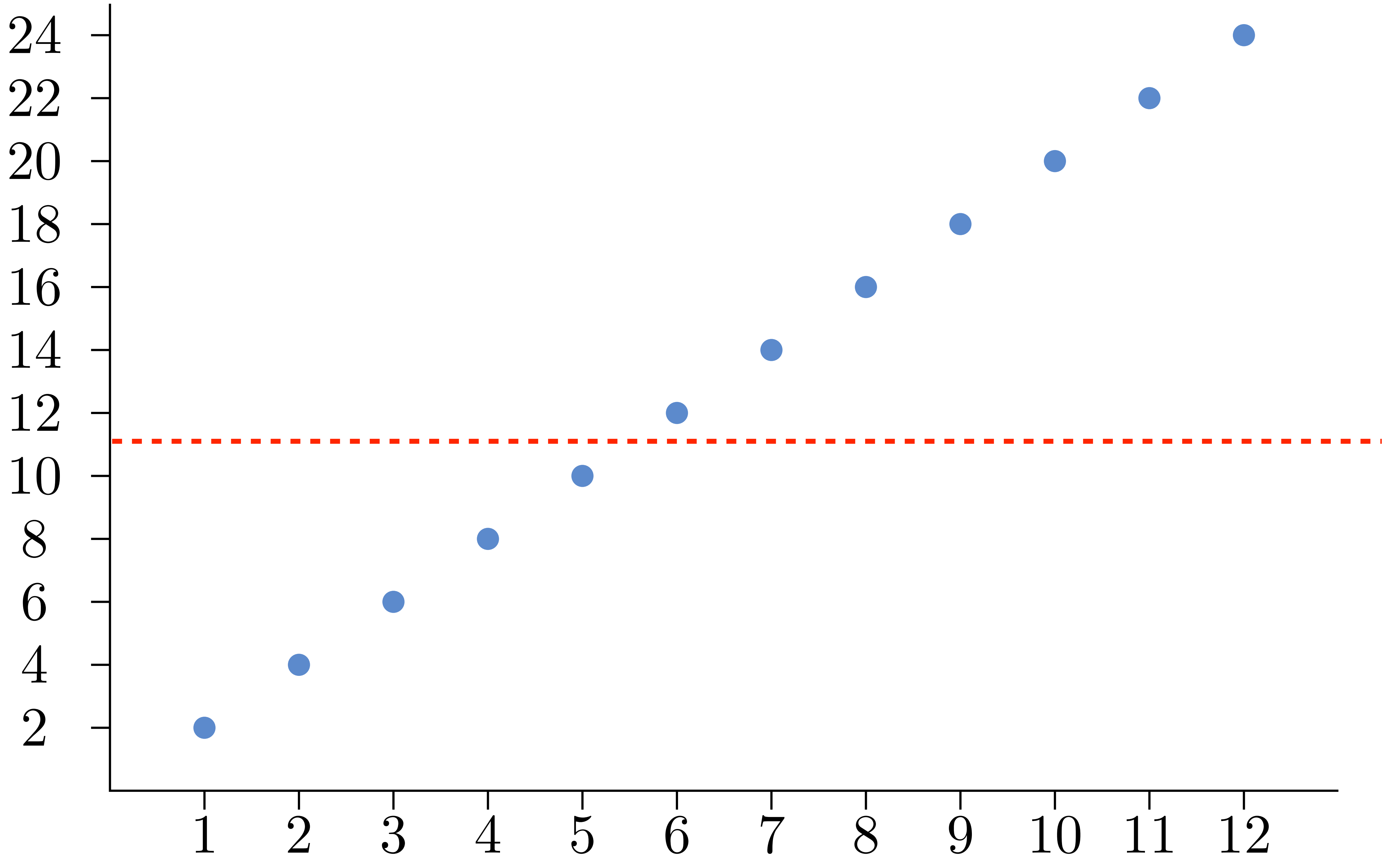

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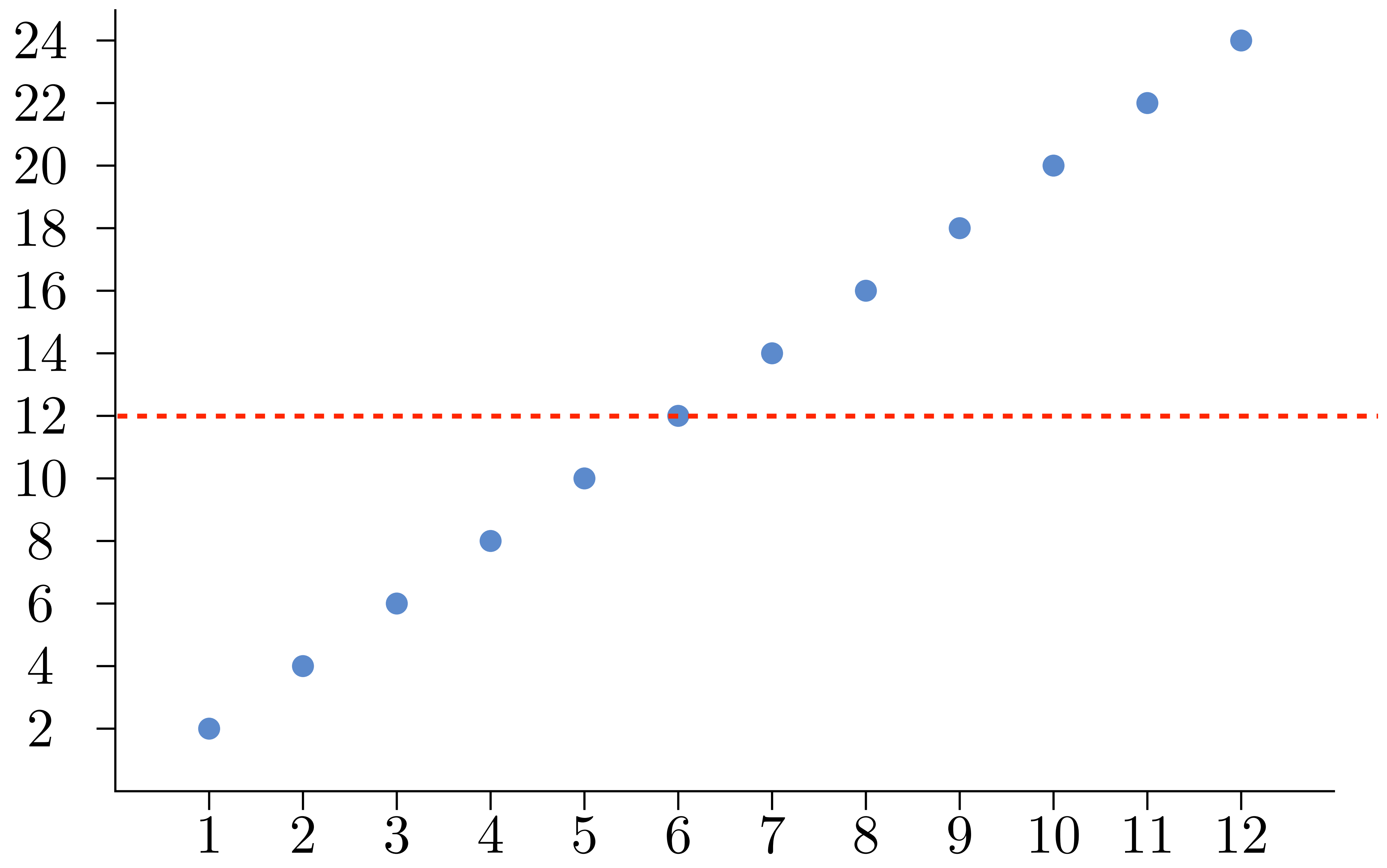

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Construct the sequences $f(n) + n$ and $f^{\downarrow}(n) + n$.

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$f(n) + n$	3	6	9	12	15	18	21	24	27	30	33	36

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$f(n) + n$	3	6	9	12	15	18	21	24	27	30	33	36
$f^{\downarrow}(n) + n$	1	2	4	5	7	8	10	11	13	14	16	17

Theorem [Lambek/Moser]. Given an increasing integer sequence $f(n)$, the two integer sequences $f(n) + n$ and $f^{\downarrow}(n) + n$ are complementary.

Theorem [Lambek/Moser]. Given an increasing integer sequence $f(n)$, the two integer sequences $f(n) + n$ and $f^{\downarrow}(n) + n$ are complementary.

Try it yourself with the increasing sequence $f(n) = n^2$

or

your favorite increasing integer sequence!

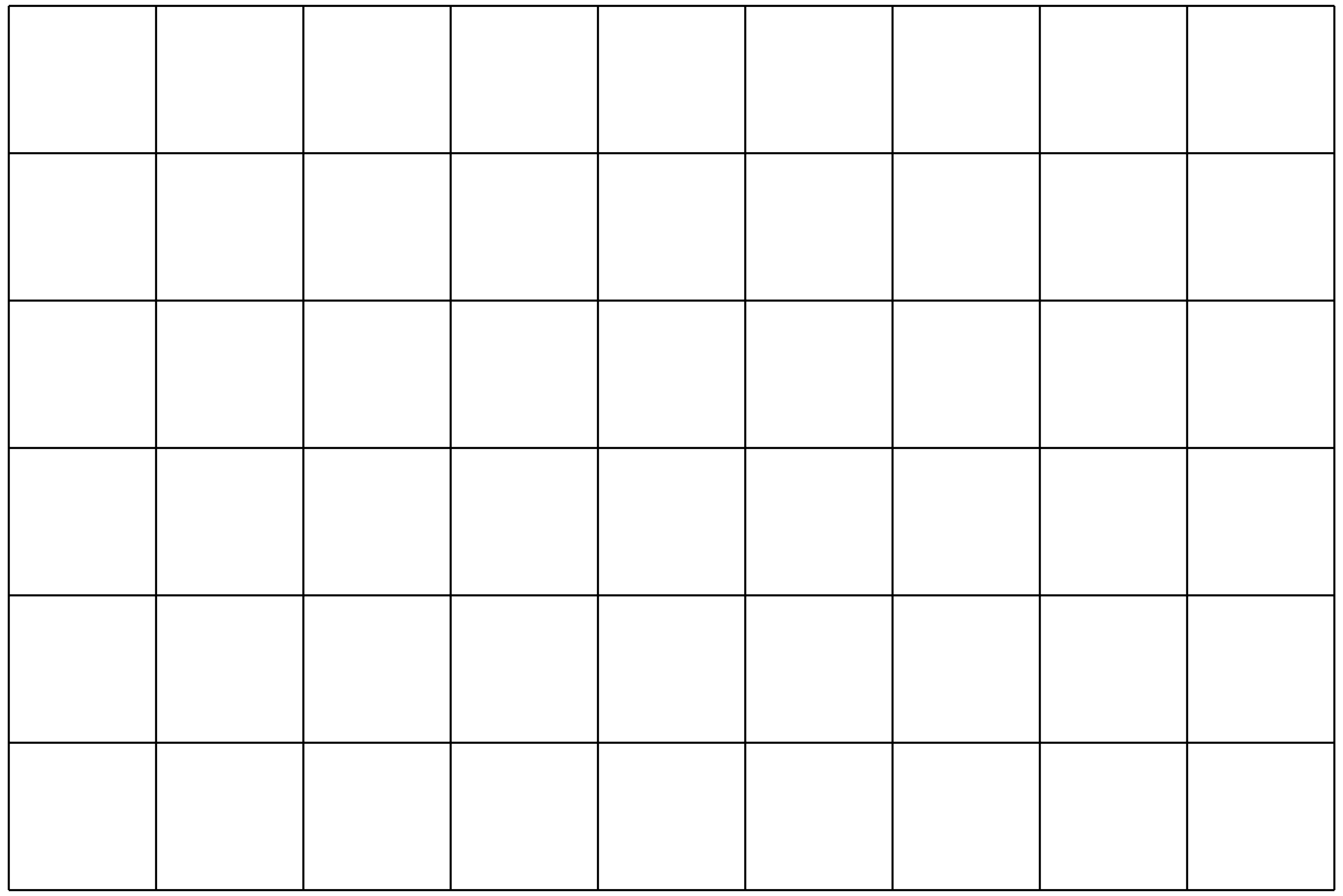
Bonus: Beatty Sequences and a Game

Wythoff's game (equivalent version)

Bonus: Beatty Sequences and a Game

Wythoff's game (equivalent version)

Players alternate moving queen on $m \times n$ board:



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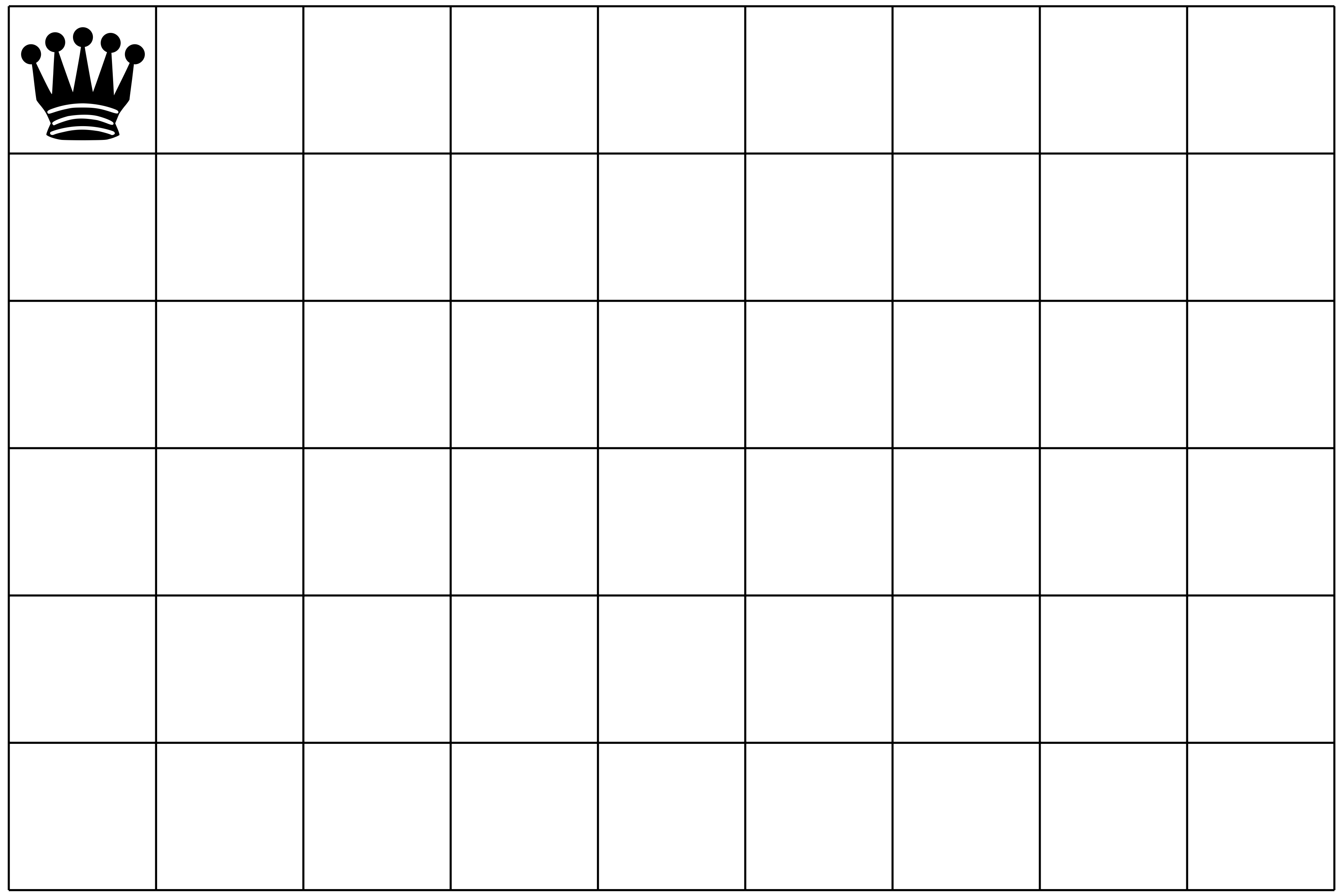

Valid moves:

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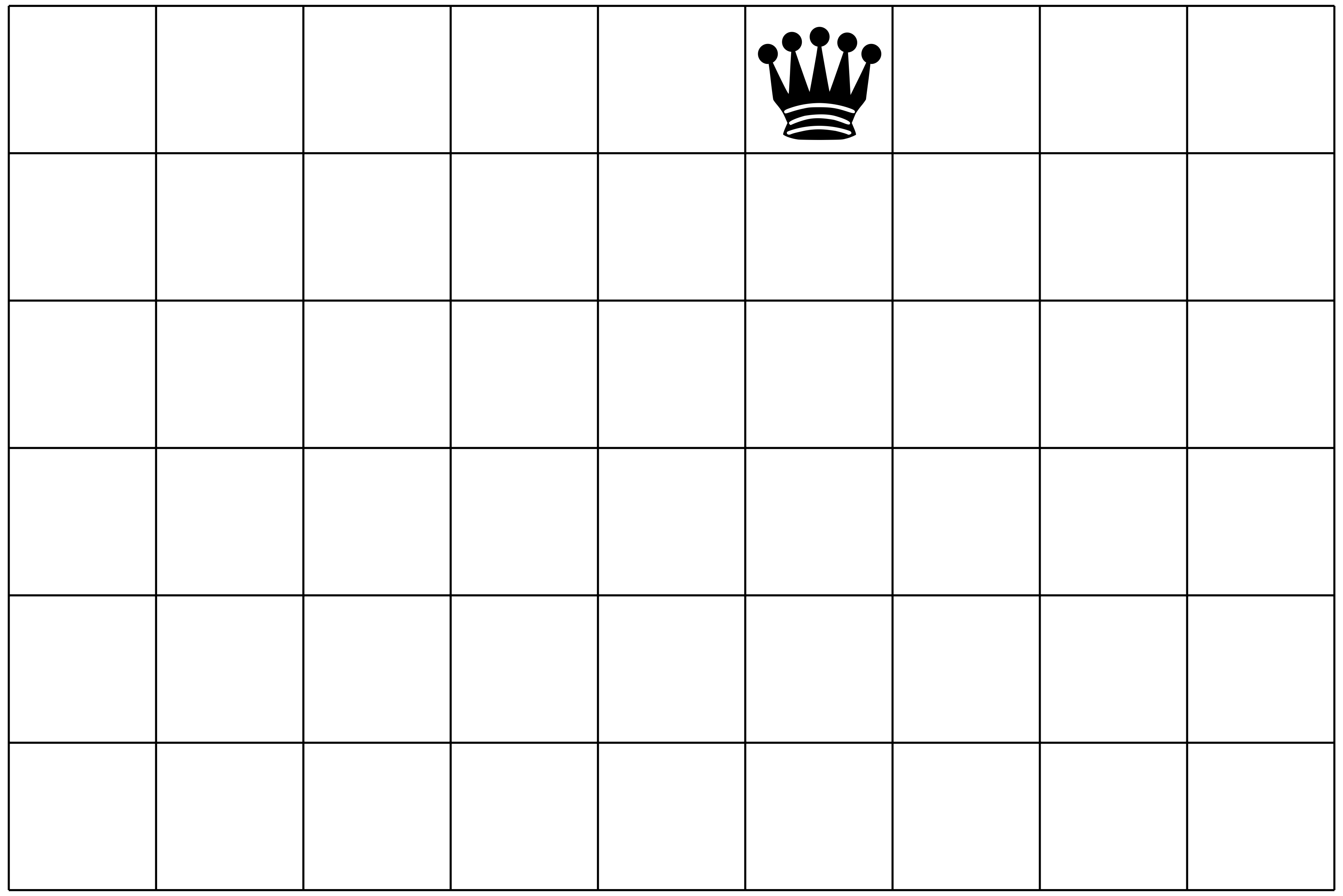

Valid moves:

1. Any number of spaces right

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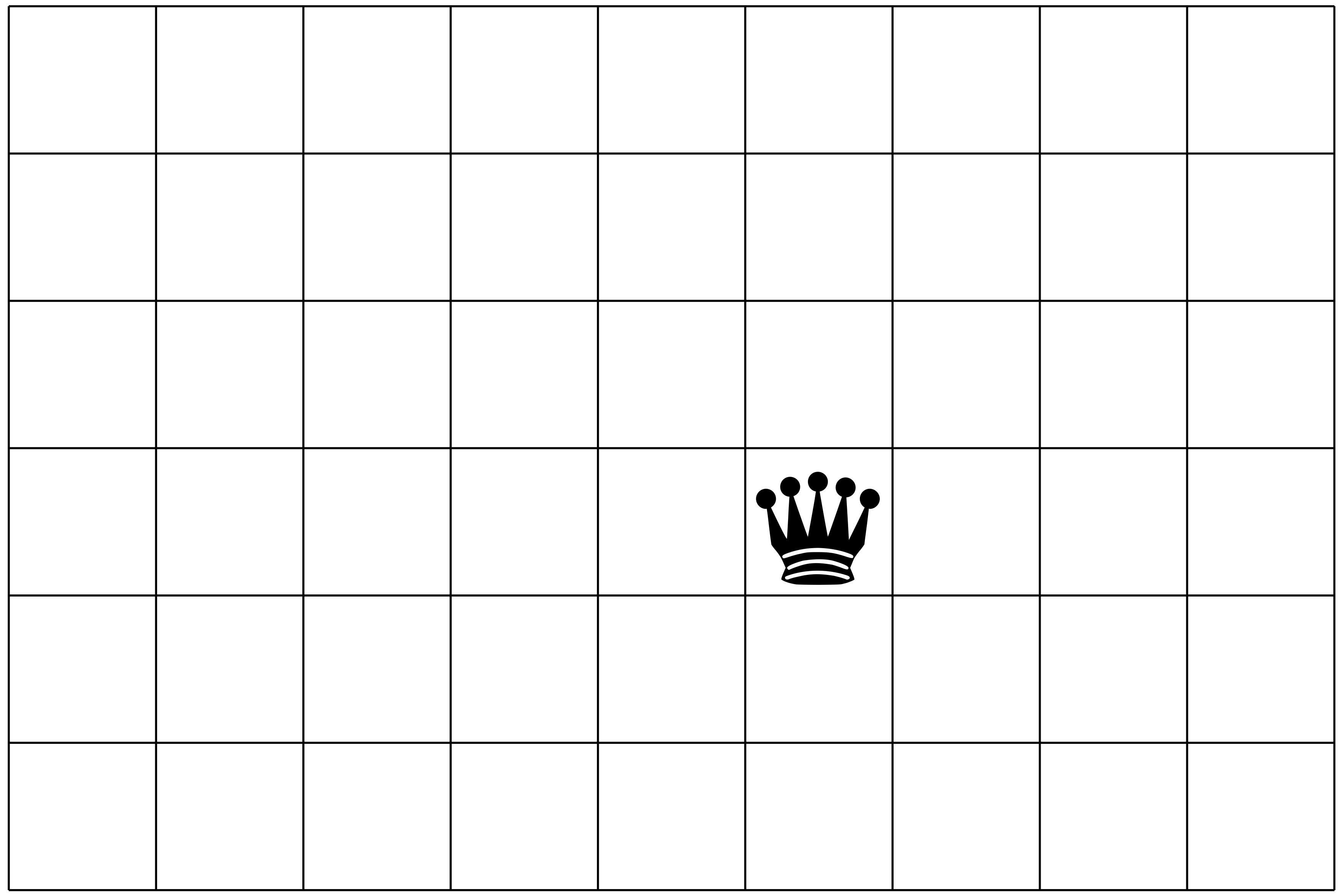

Valid moves:

1. Any number of spaces right

2. Any number of spaces down

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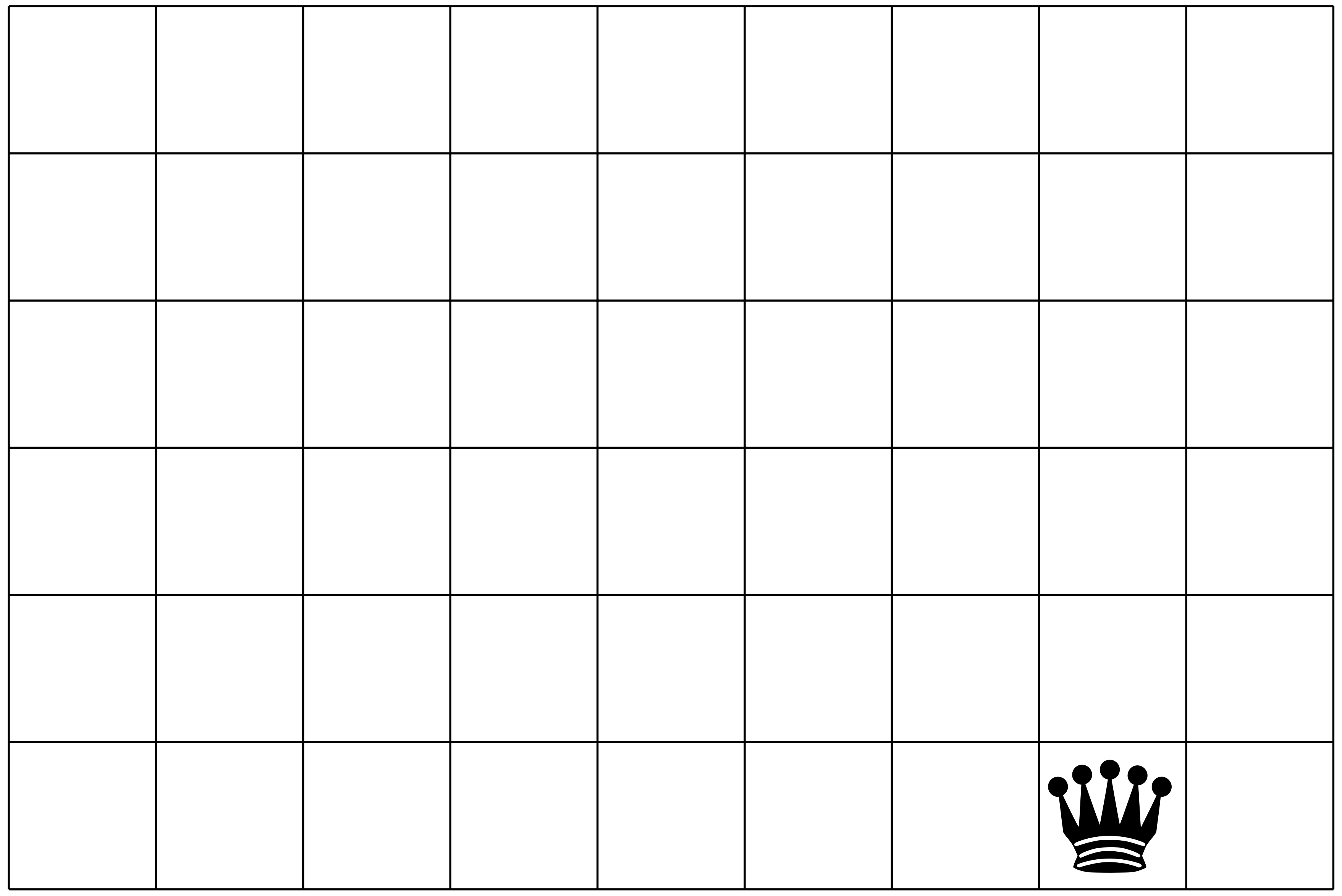

Valid moves:

1. Any number of spaces right
2. Any number of spaces down
3. Any number of spaces SE diagonal

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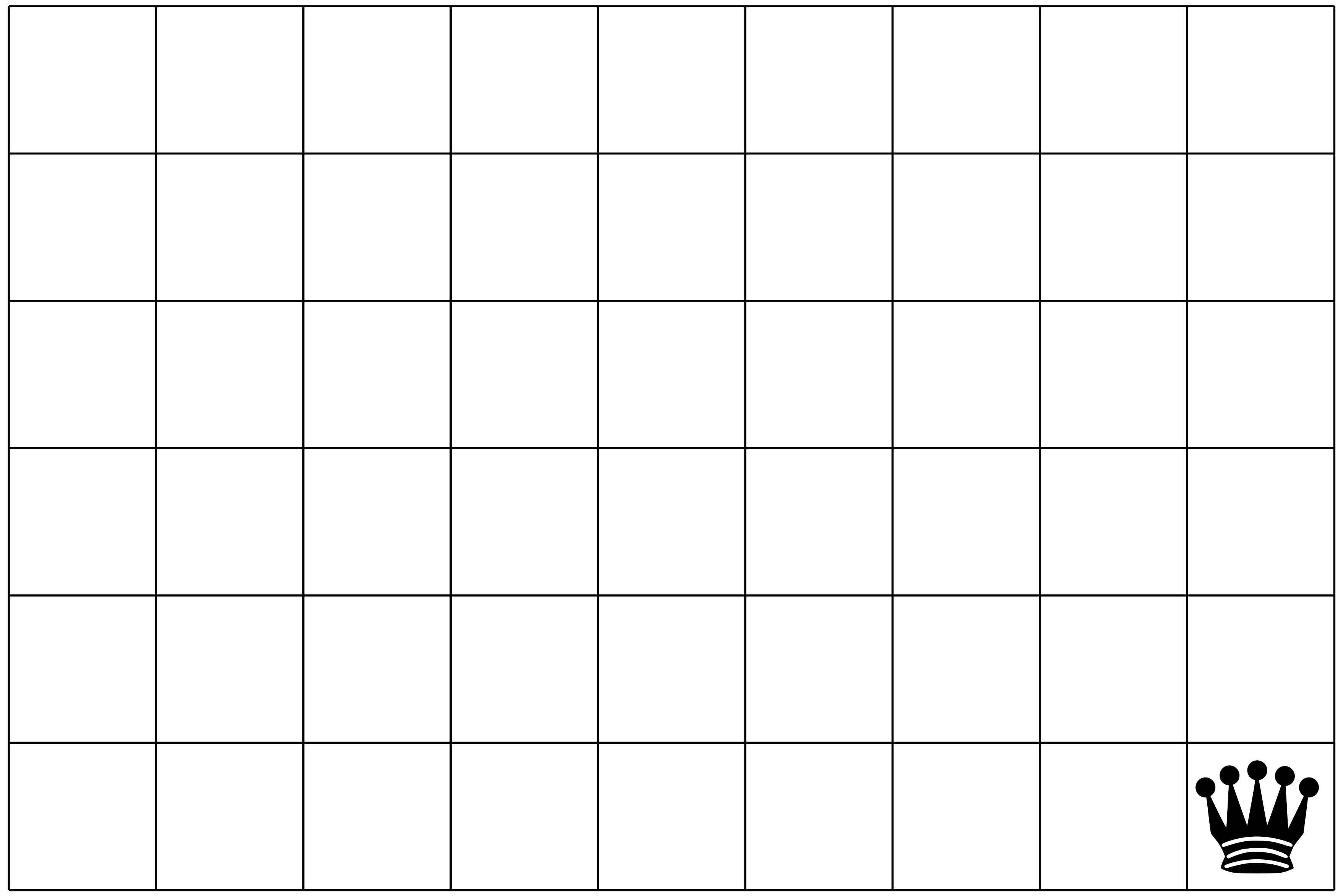


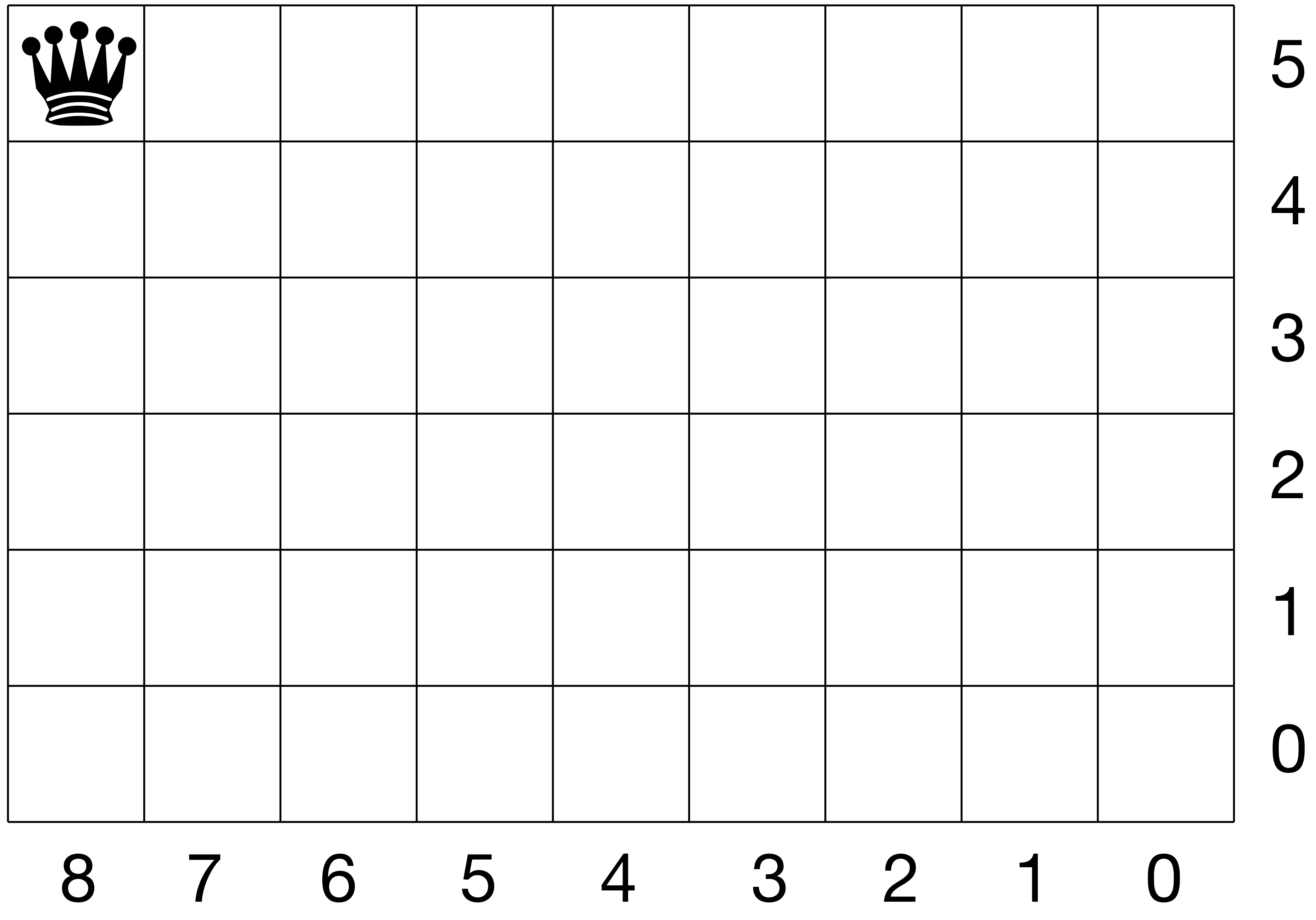
Valid moves:

1. Any number of spaces right
2. Any number of spaces down
3. Any number of spaces SE diagonal

Player who moves queen to bottom right square wins!

Bonus: Beatty Sequences and a Game

Wythoff's game strategy:



Bonus: Beatty Sequences and a Game

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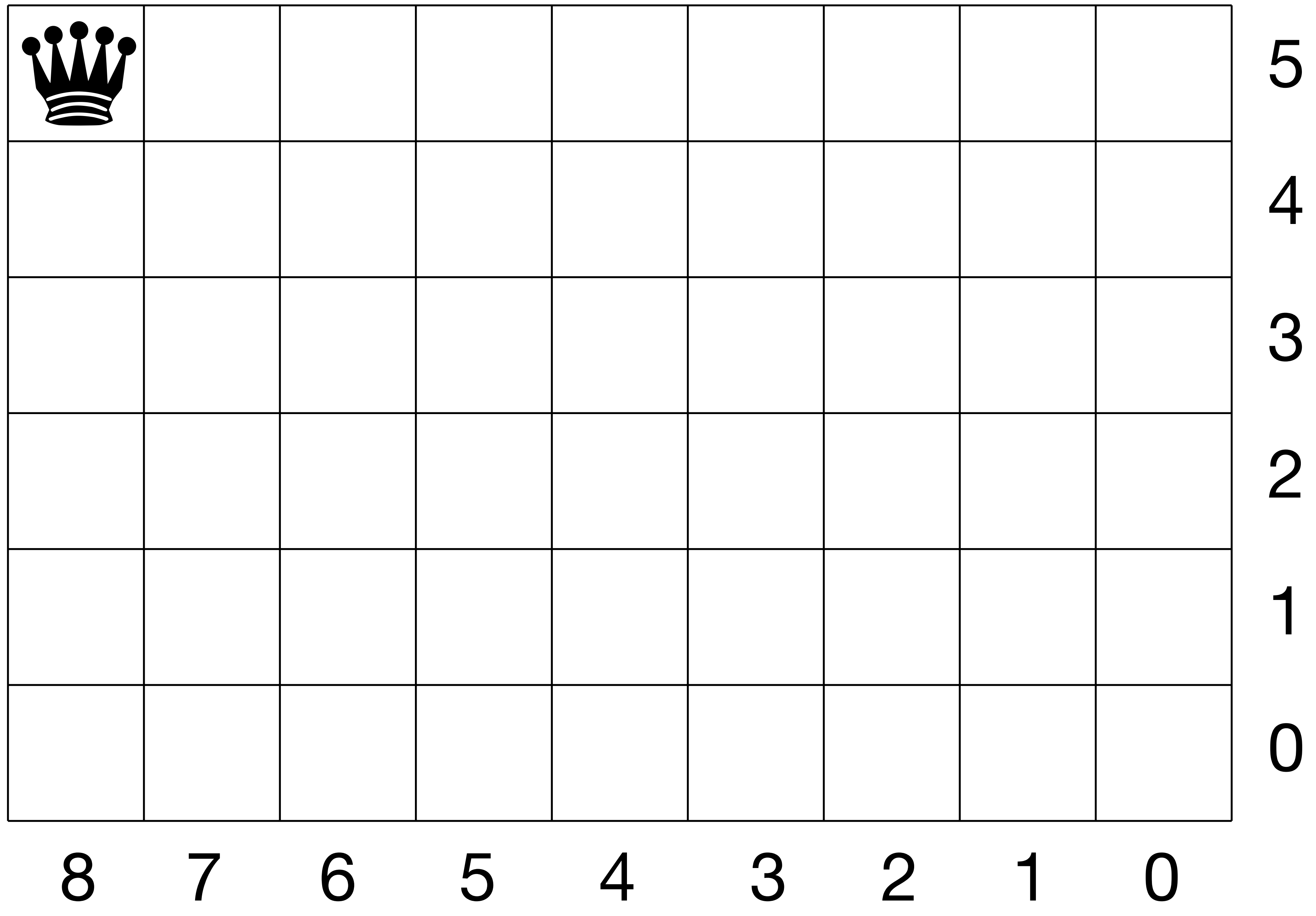

$$\left(\lfloor n \cdot \phi \rfloor\right)_{n=0}^{\infty} = \quad (0, 1, 3, 4, 6, 8, 9, ...)$$

$$\left(\lfloor n \cdot \phi/(\phi - 1) \rfloor\right)_{n=0}^{\infty} = \quad (0, 2, 5, 7, 10, 13, 15, ...)$$

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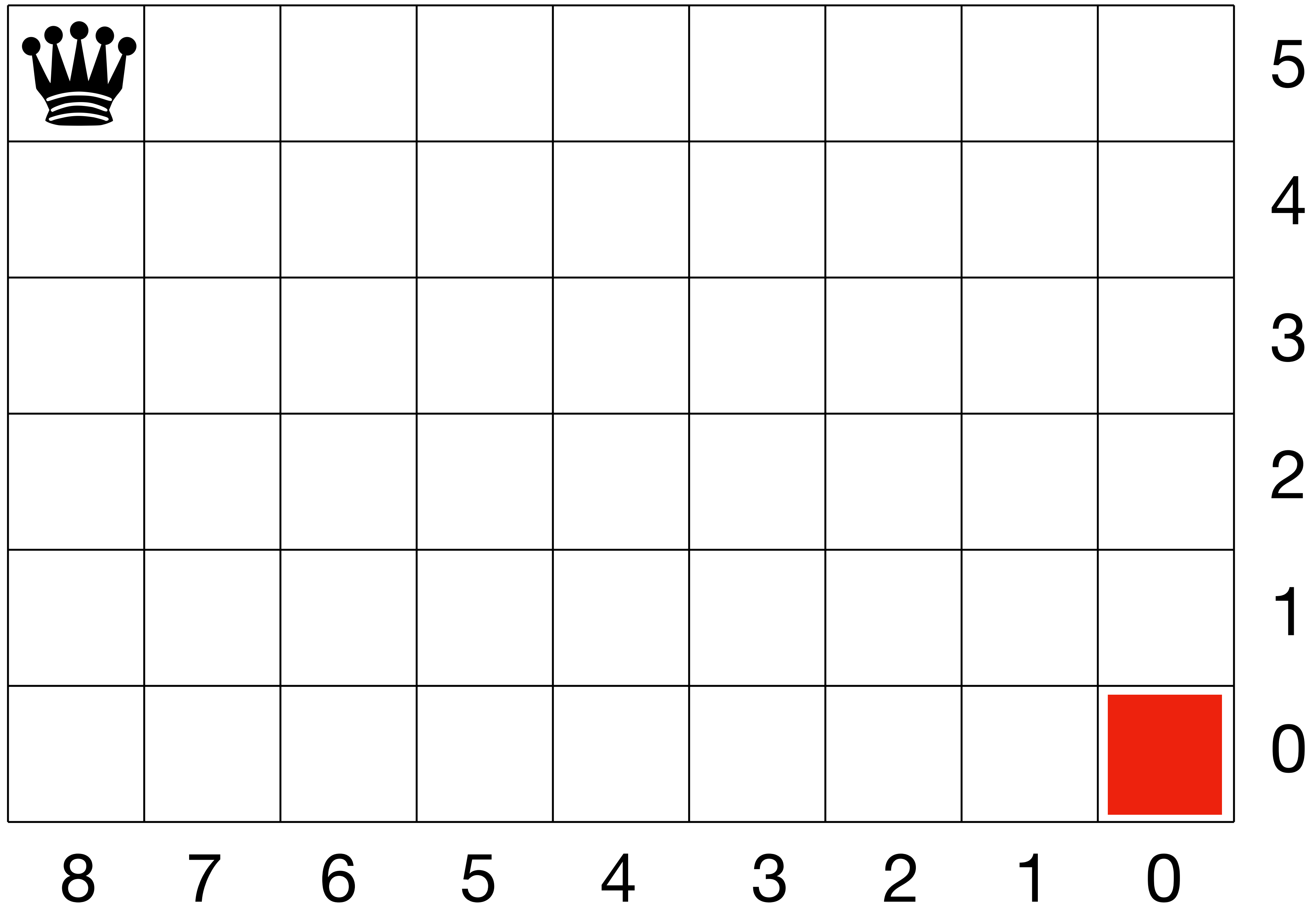

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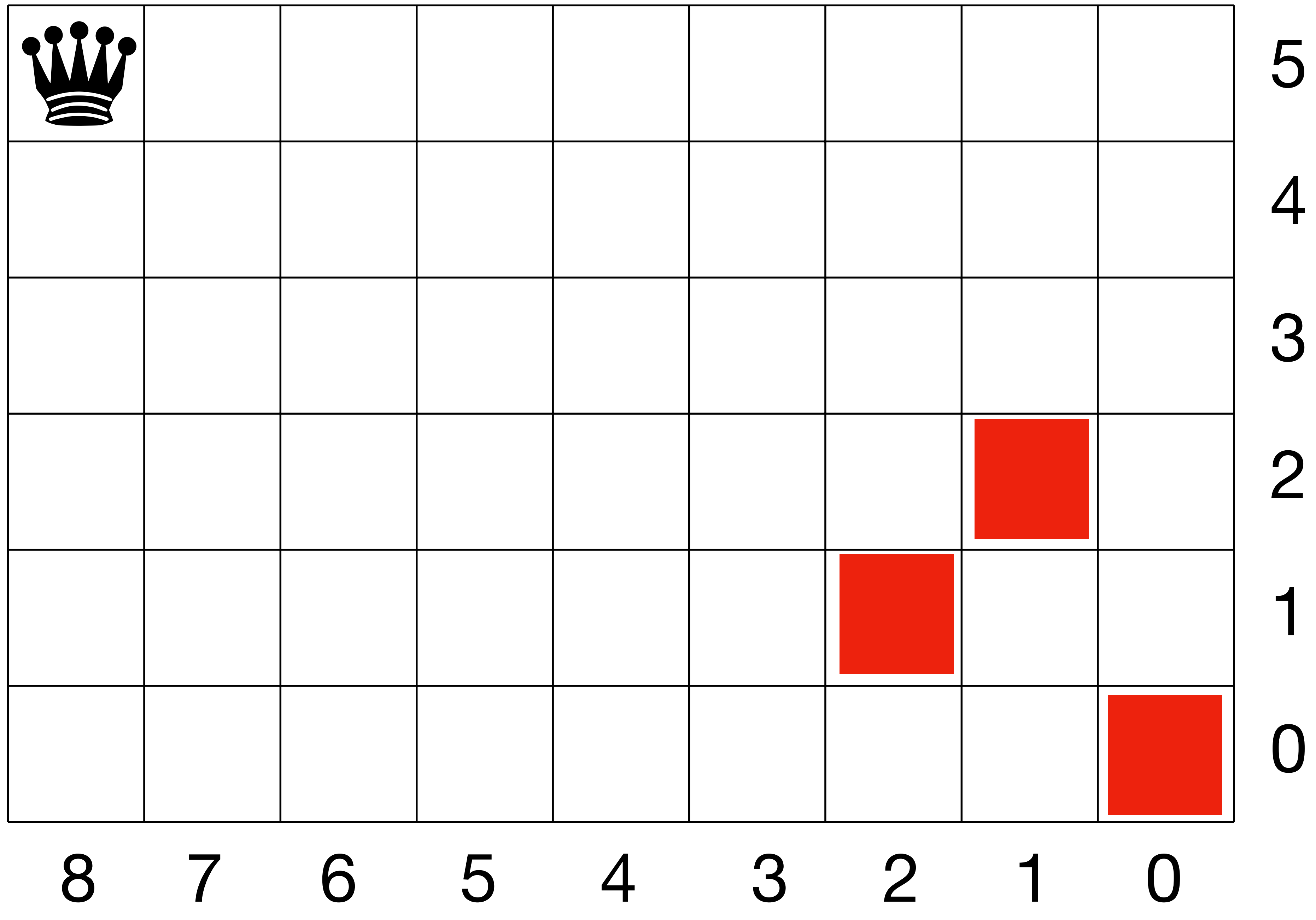

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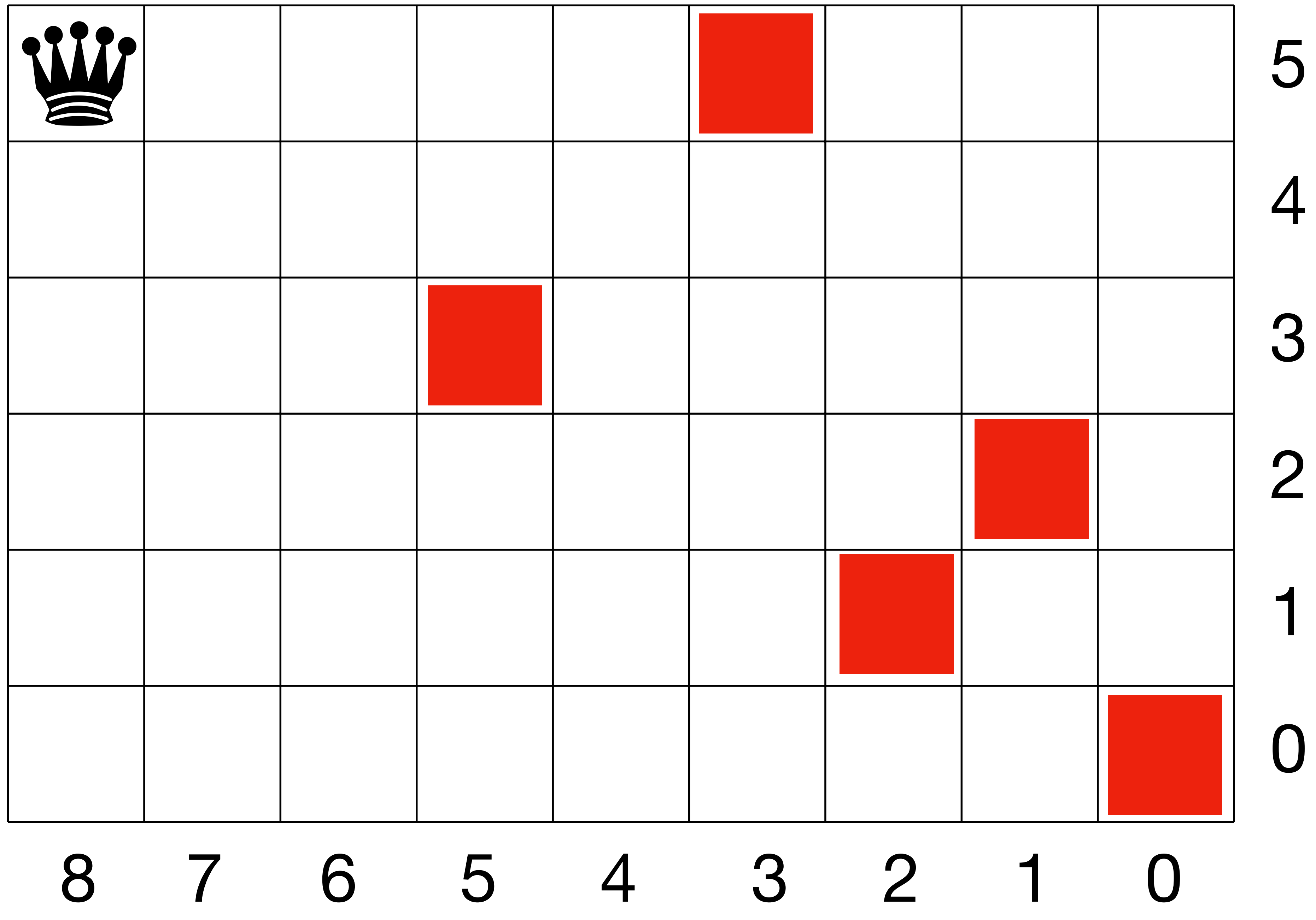

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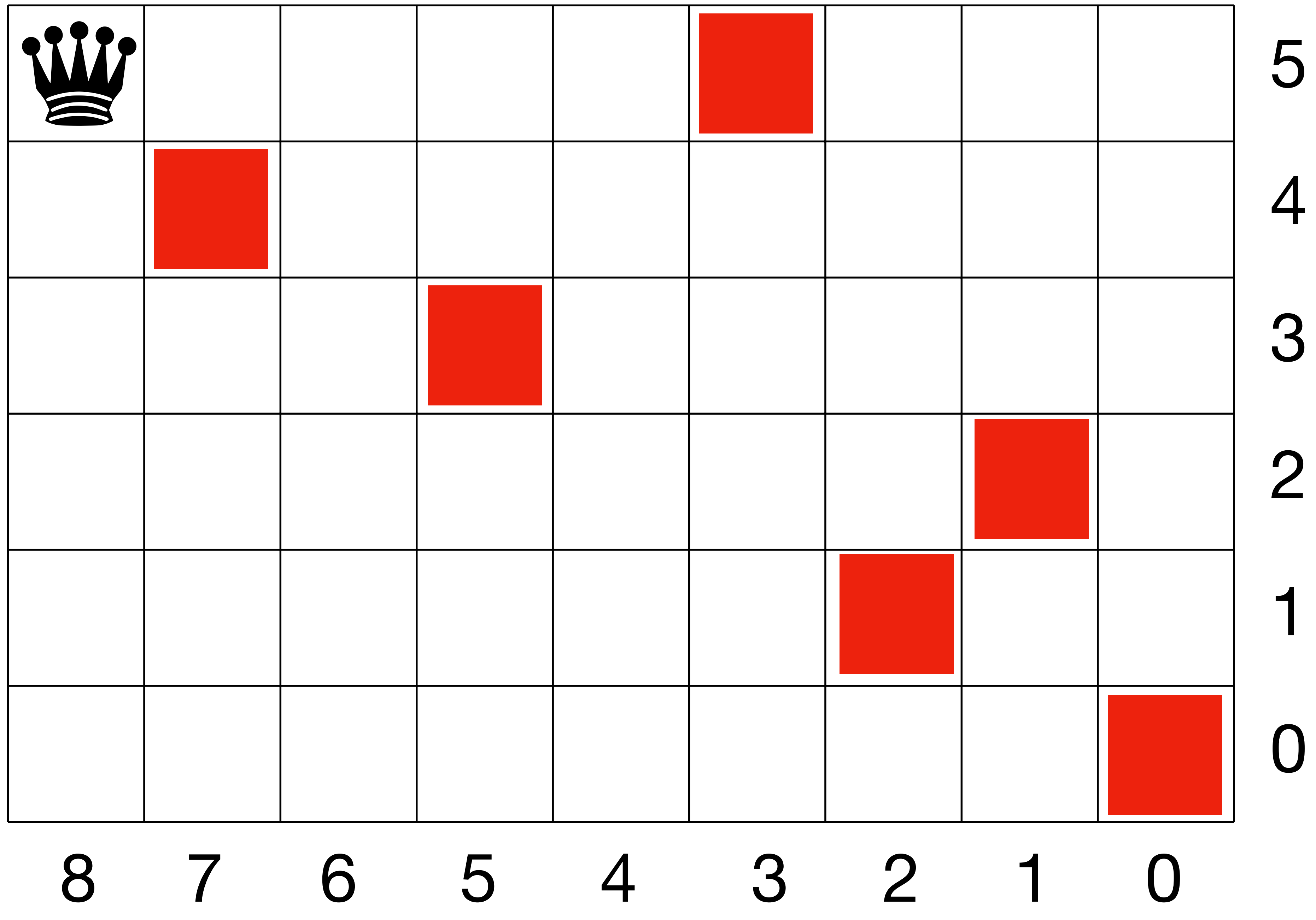

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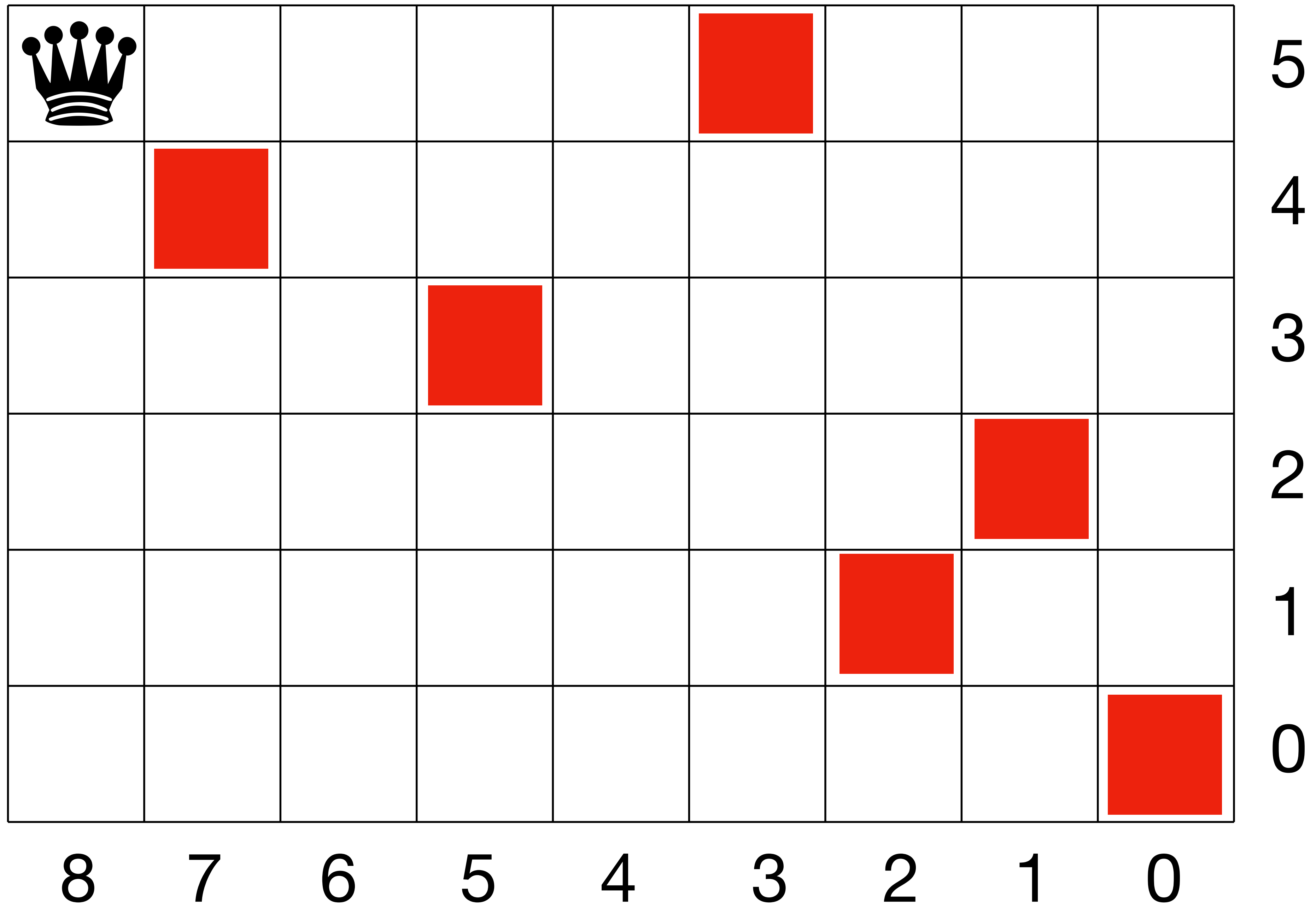

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Strategy: move to red square

Bonus: Beatty Sequences and a Game

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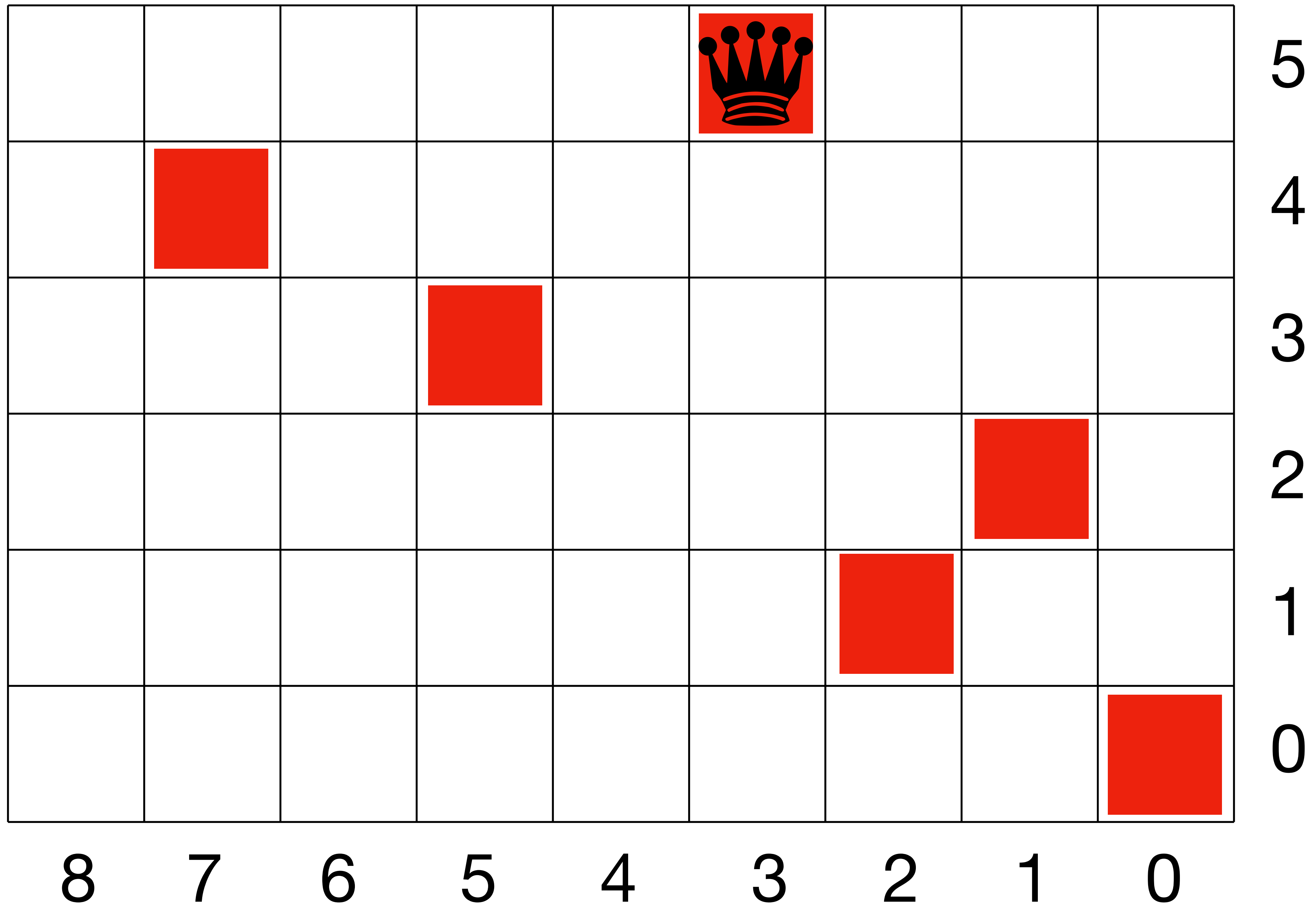

Player 1

$$\left(\lfloor n \cdot \phi \rfloor\right)_{n=0}^{\infty} = \quad (0, 1, 3, 4, 6, 8, 9, \ldots)$$

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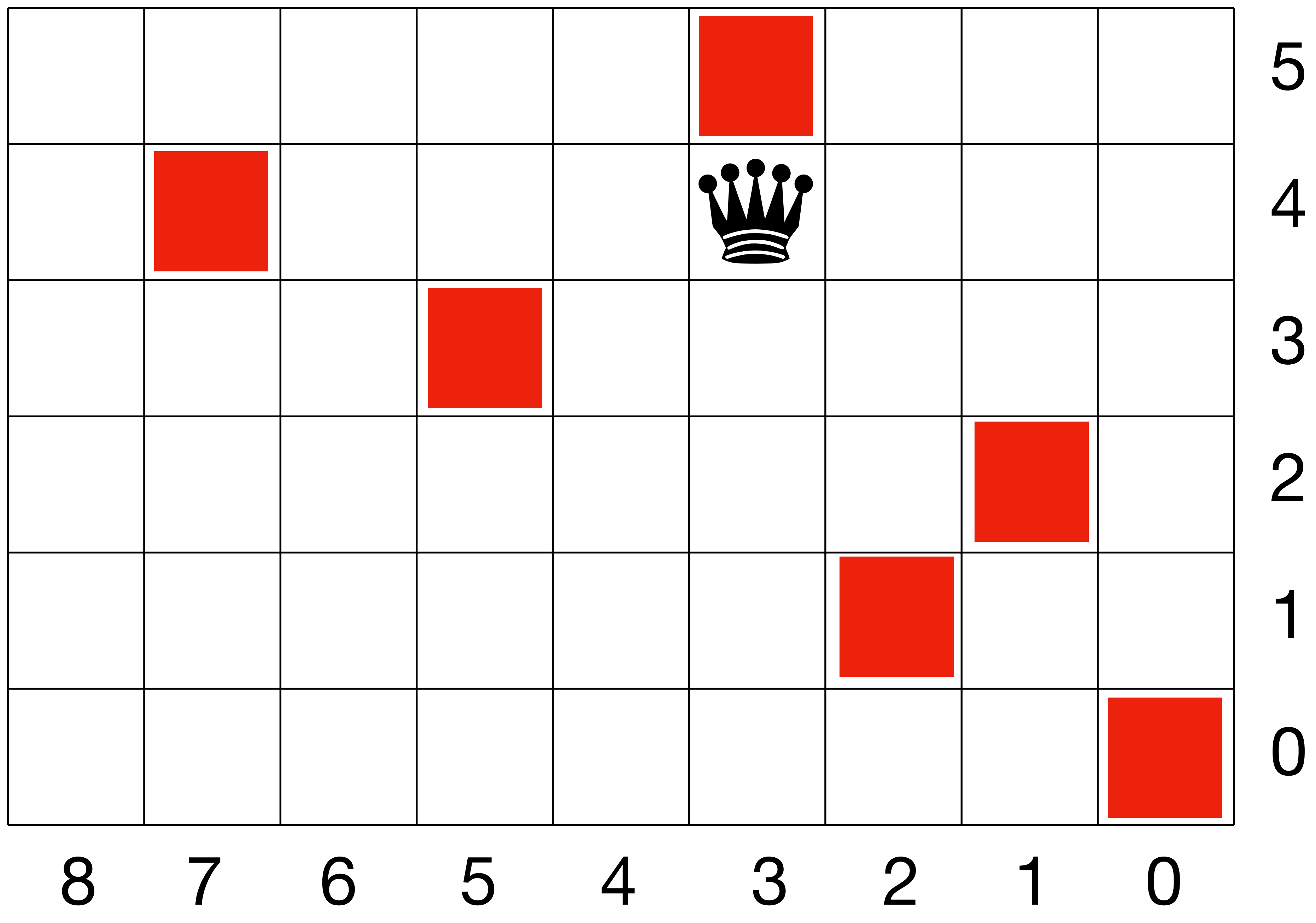

Player 2

$$\left(\lfloor n \cdot \phi \rfloor\right)_{n=0}^{\infty} = \quad (0, 1, 3, 4, 6, 8, 9, \ldots)$$

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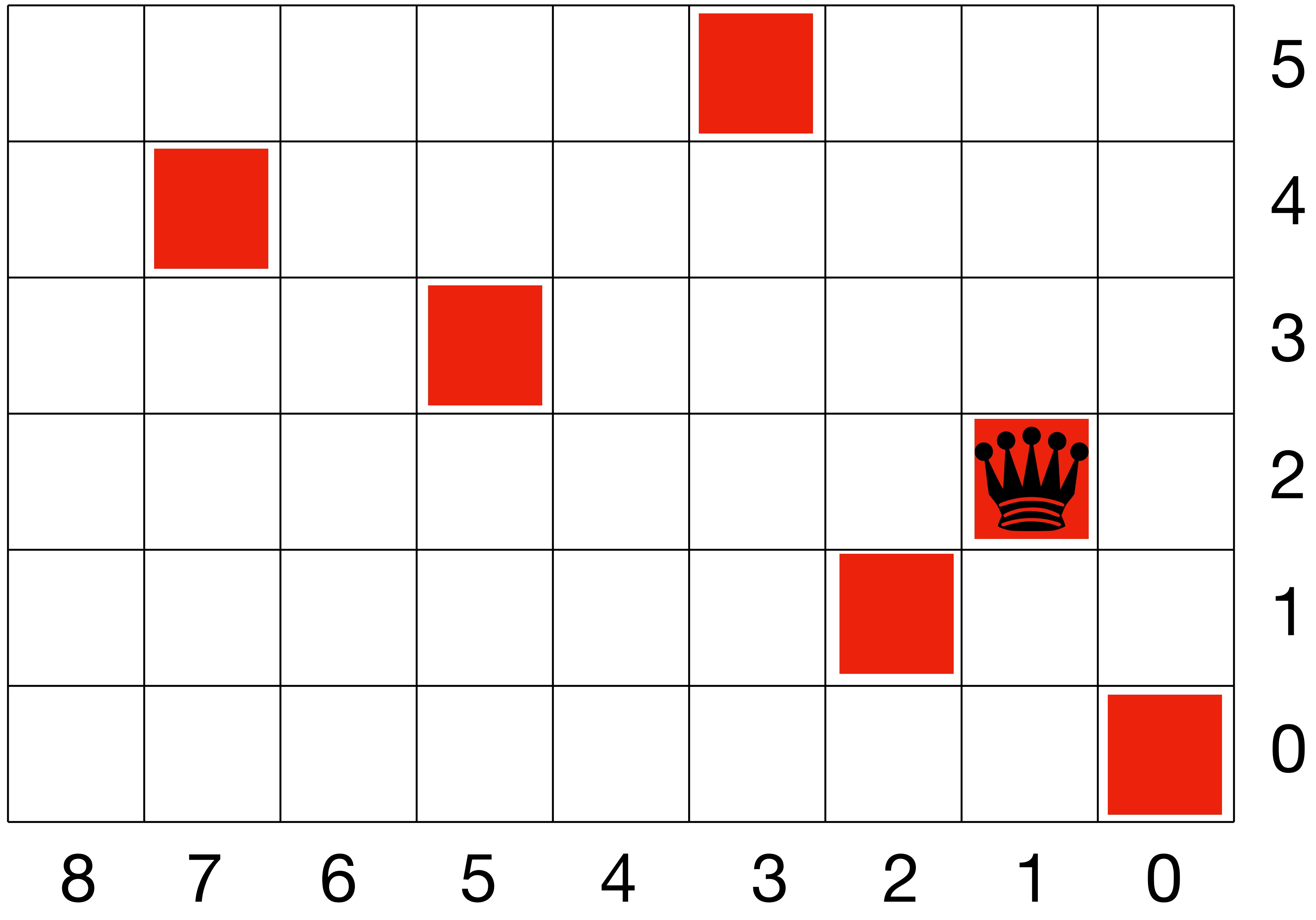

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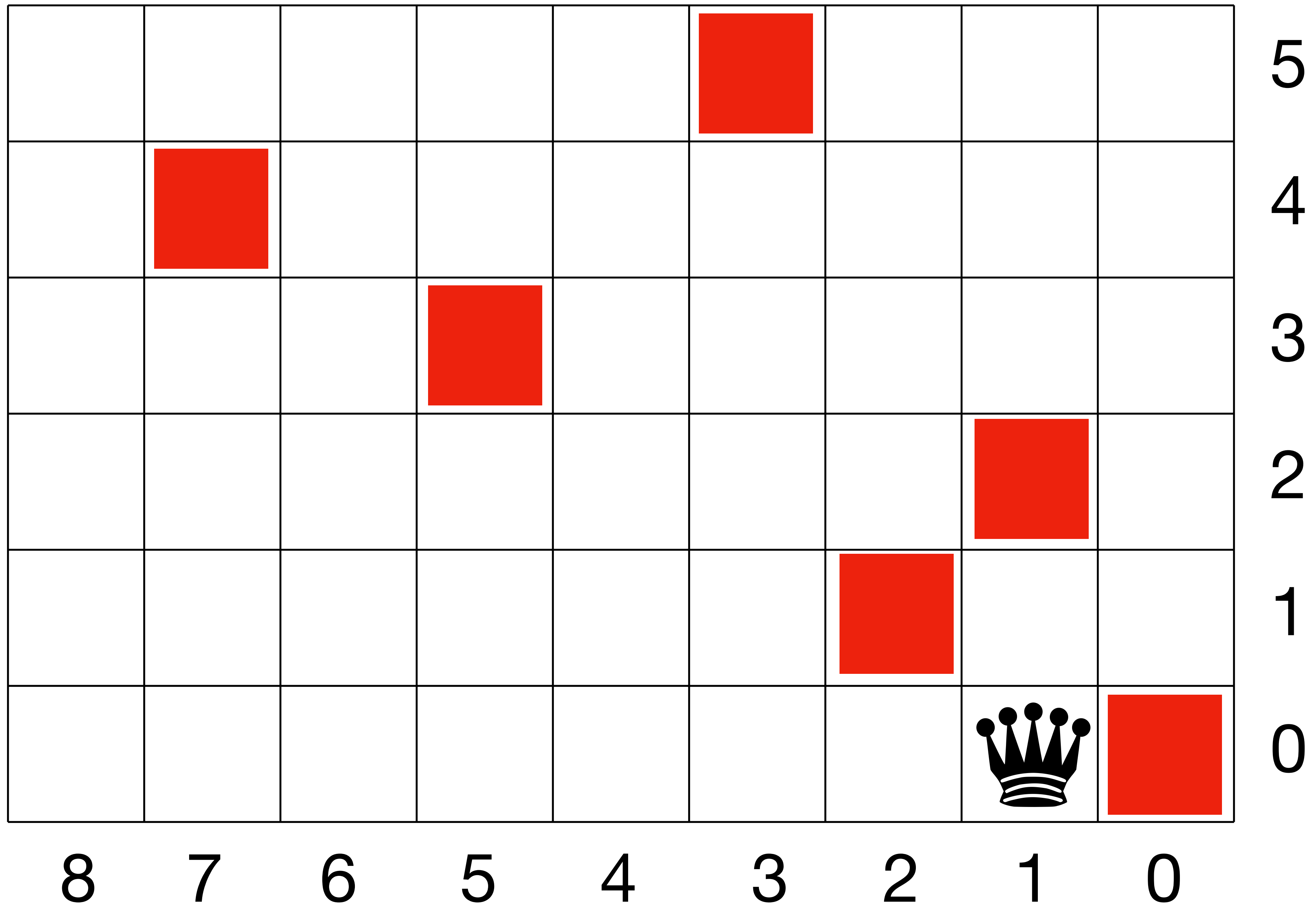

Player 2

$$\left(\lfloor n \cdot \phi \rfloor\right)_{n=0}^{\infty} = \quad (0, 1, 3, 4, 6, 8, 9, \ldots)$$

Strategy: move to red square

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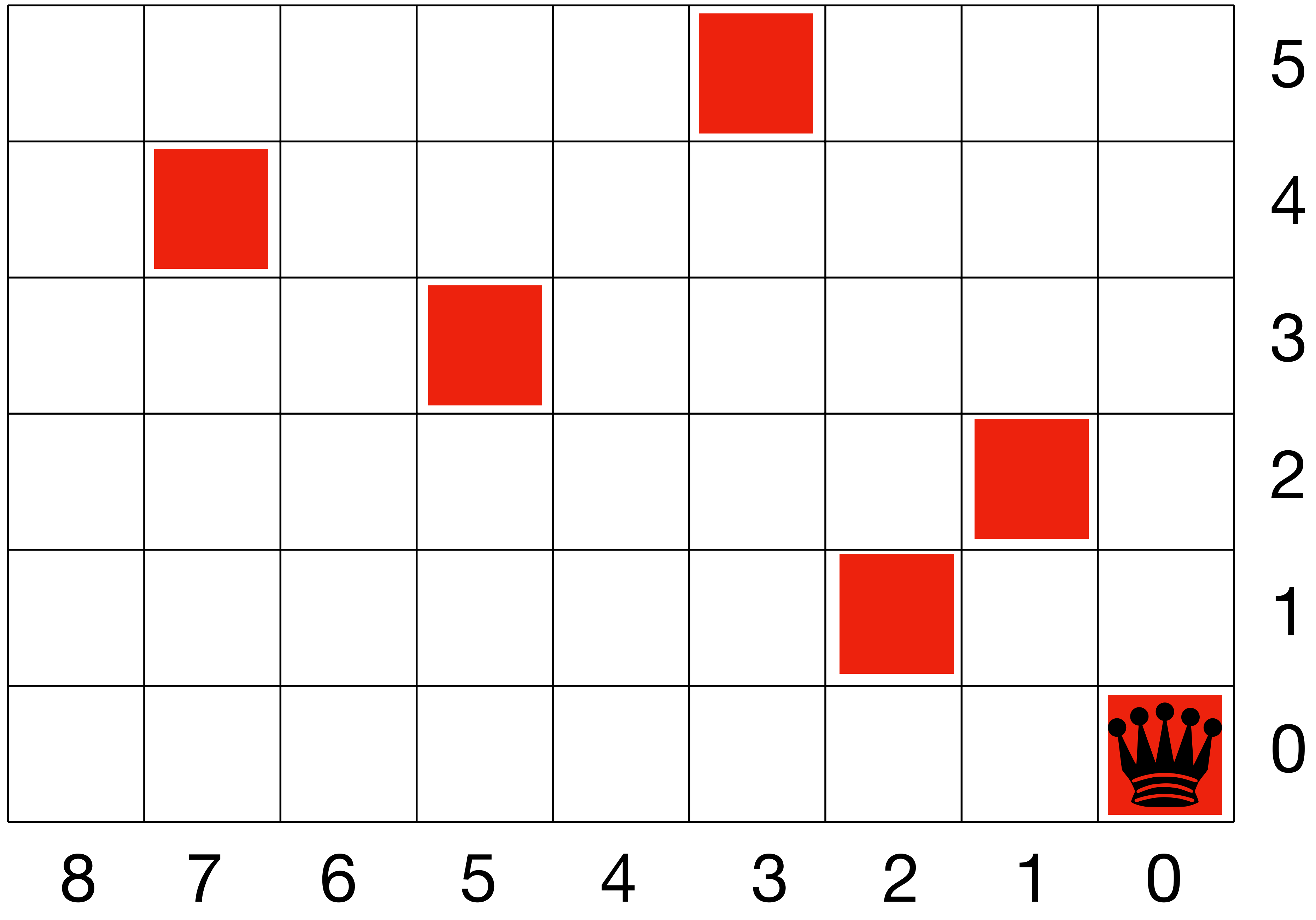

Player 1

Wins!

$$\left(\lfloor n \cdot \phi \rfloor\right)_{n=0}^{\infty} = (0, 1, 3, 4, 6, 8, 9, \ldots)$$

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Strategy: move to red square

Bonus: Rational Beatty Sequences

$$\left(\left\lfloor n \cdot \frac{3}{2} \right\rfloor\right)_{n=1}^{\infty} = (1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, \ldots)$$

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Differences:

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Differences: (2,

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Differences: (2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

Repeats 2,1 forever

Bonus: Rational Beatty Sequences

$$\left(\left\lfloor n \cdot \frac{3}{2} \right\rfloor\right)_{n=1}^{\infty} = (1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, \ldots)$$

Differences: (2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

Repeats 2,1 forever

Fact: x is rational if and only if the first difference sequence of $(\lfloor nx \rfloor)_{n=1}^{\infty}$ is periodic.

Bonus: Rational Beatty Sequences

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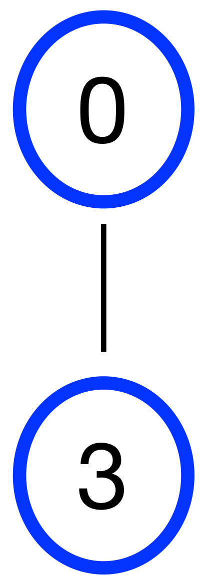

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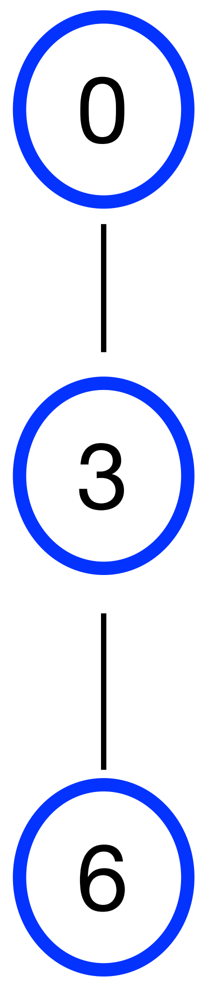

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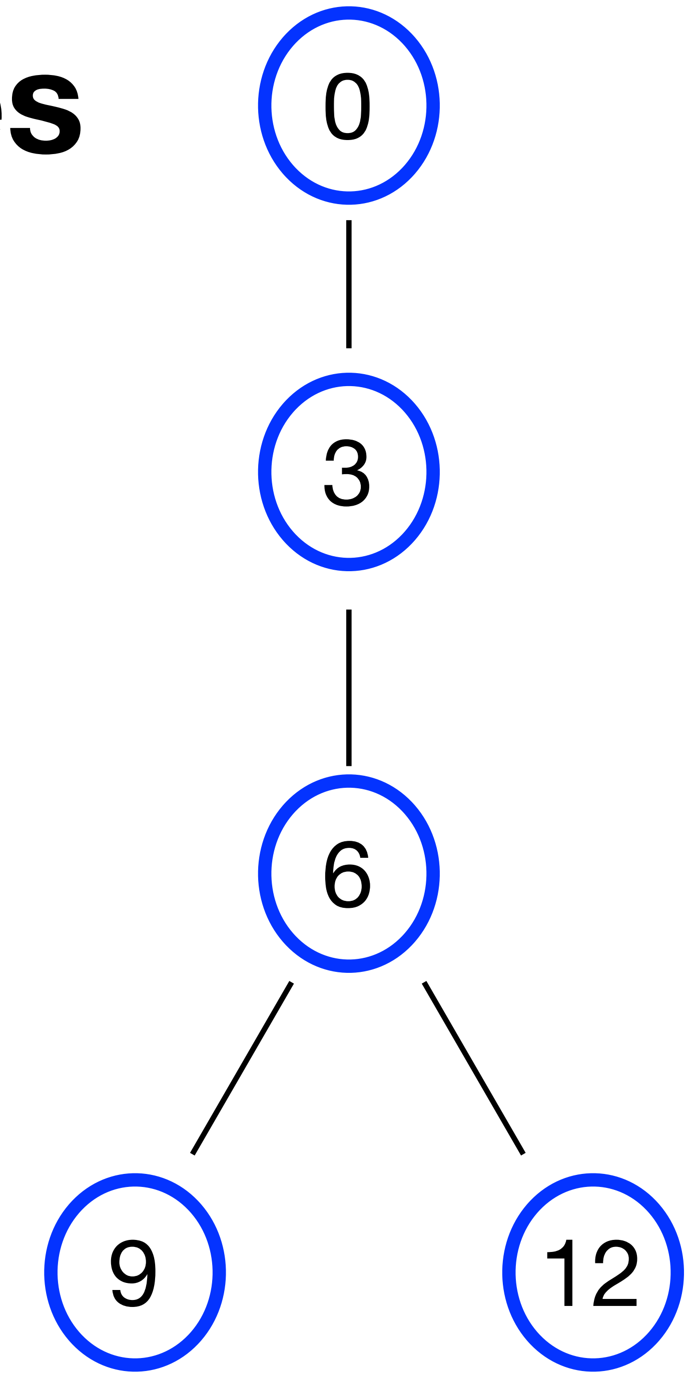

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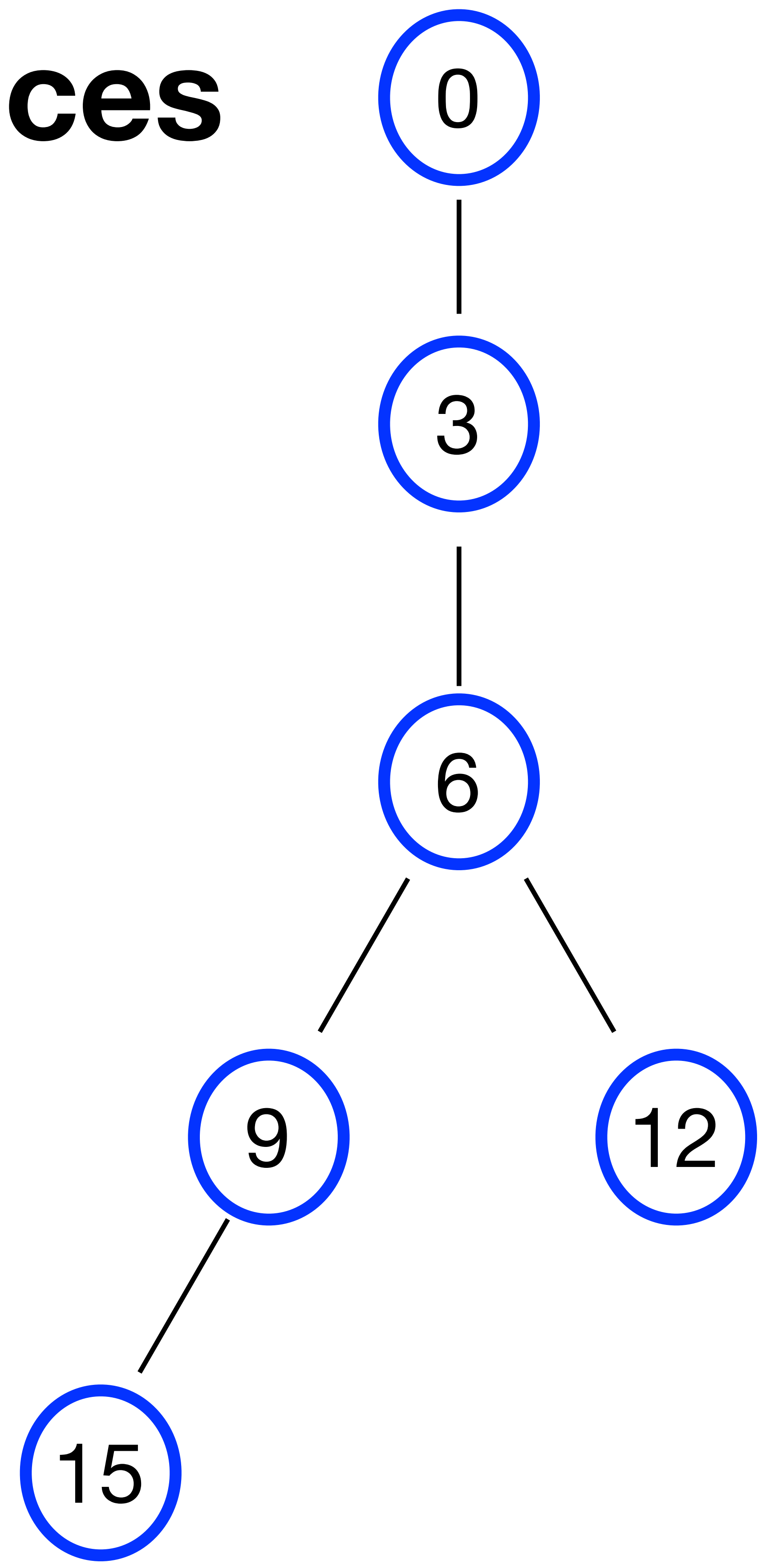

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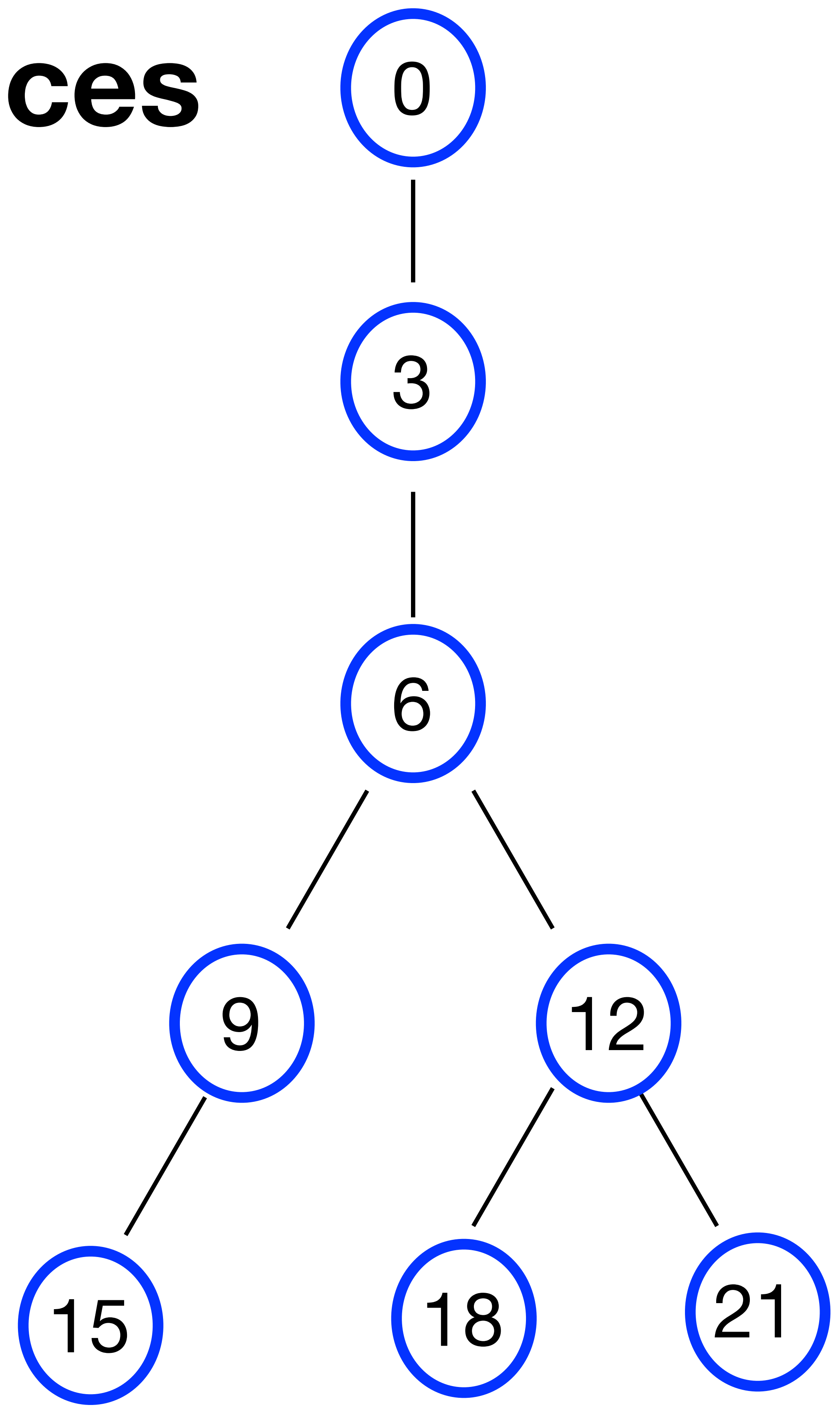

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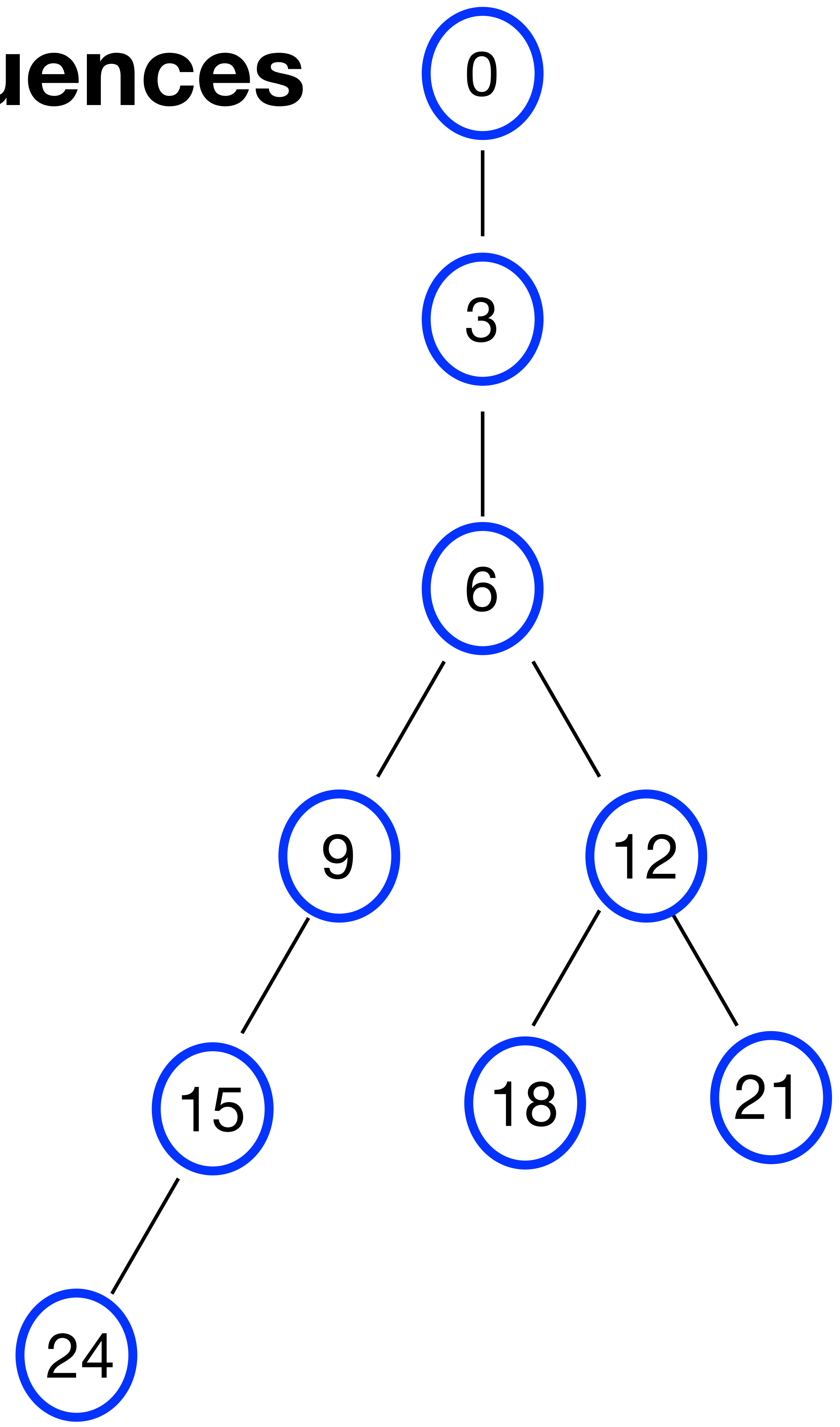

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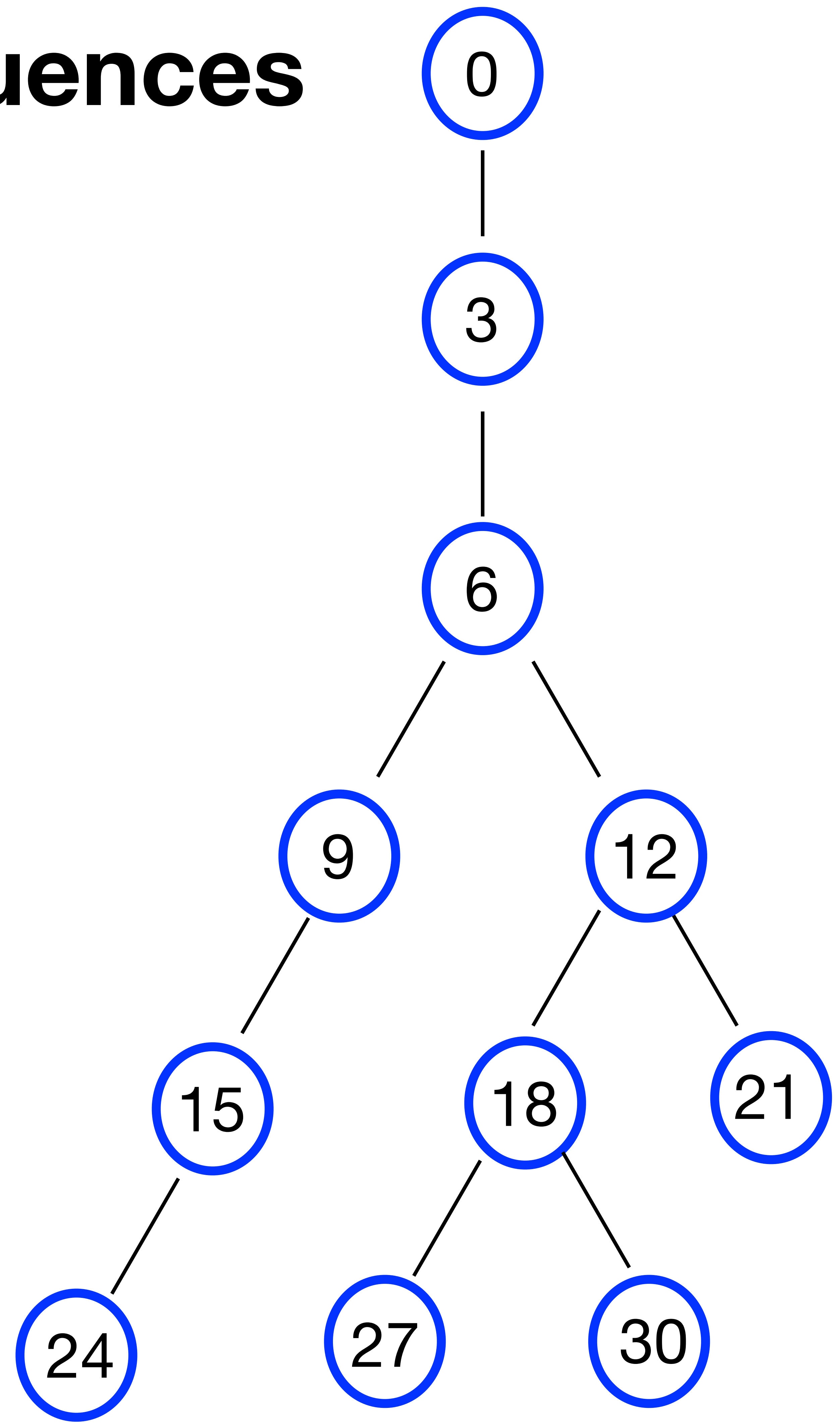

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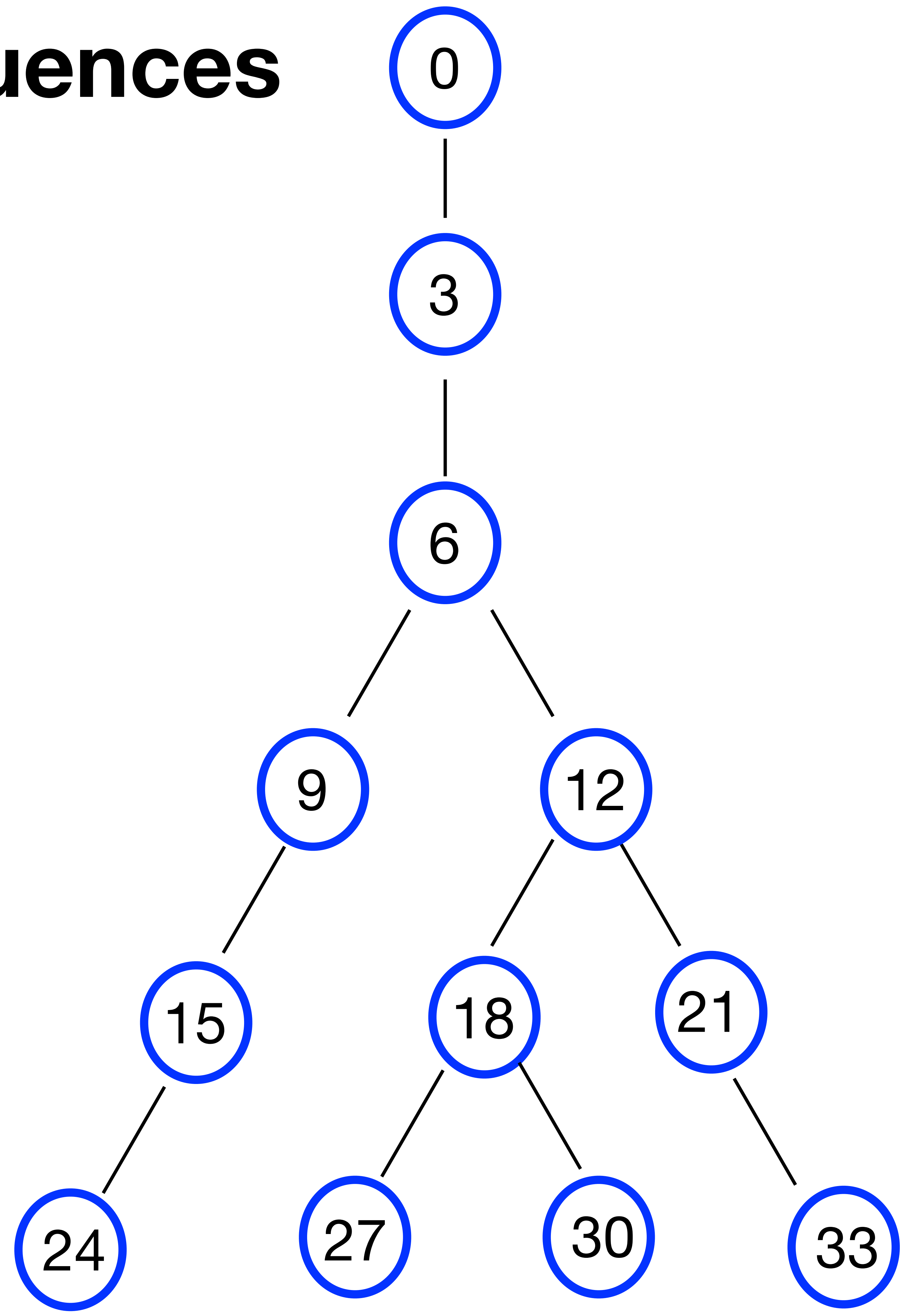

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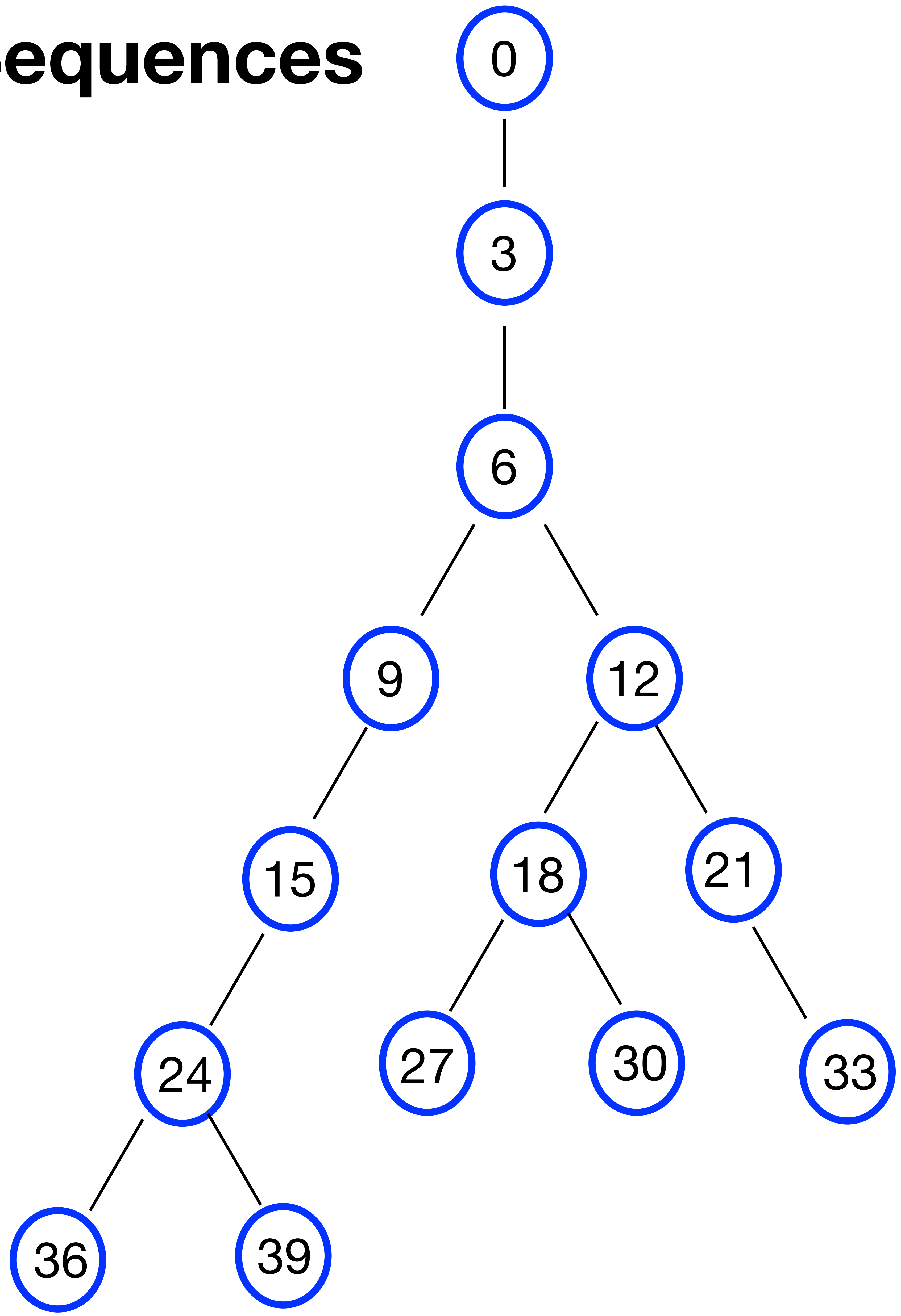

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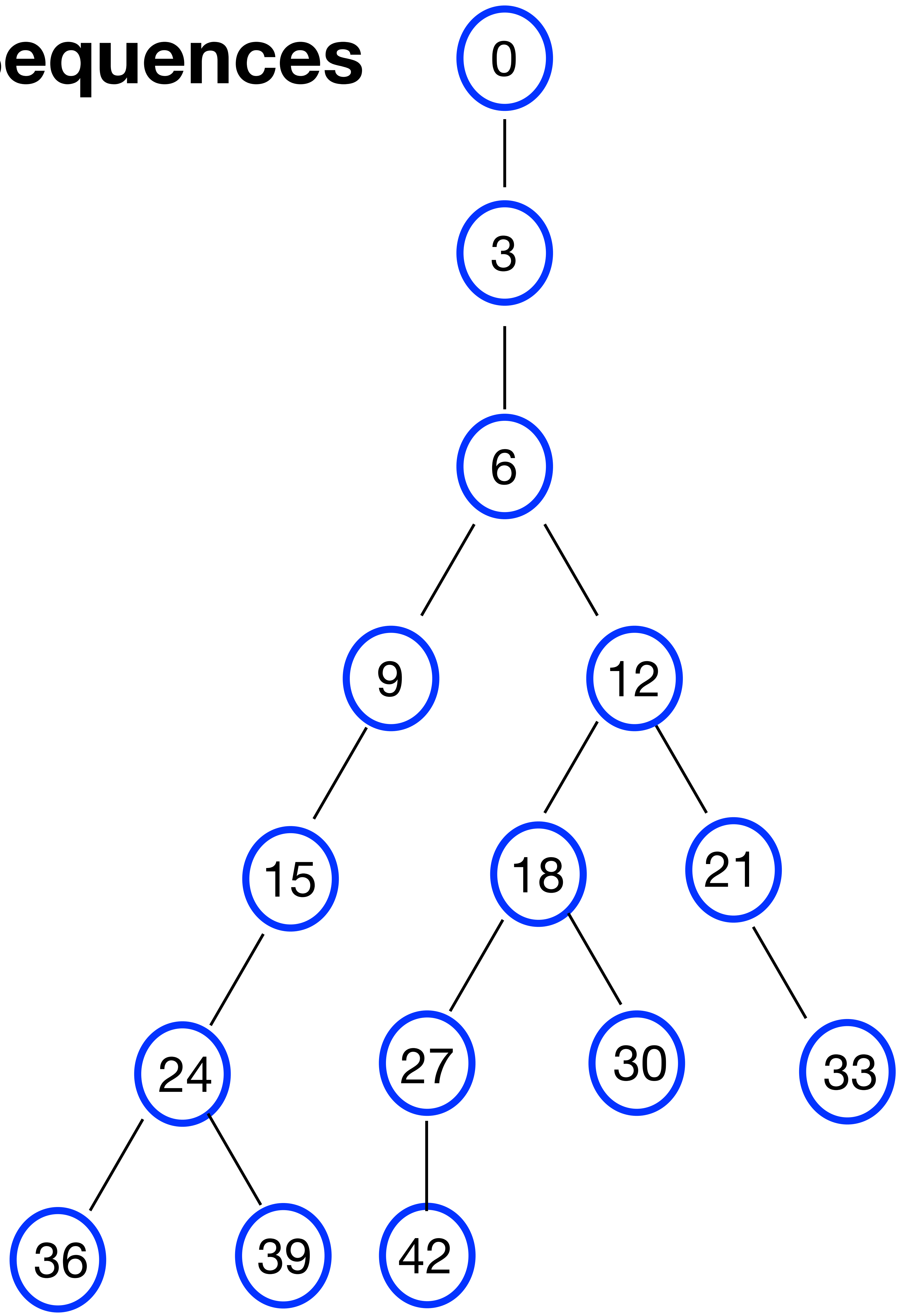

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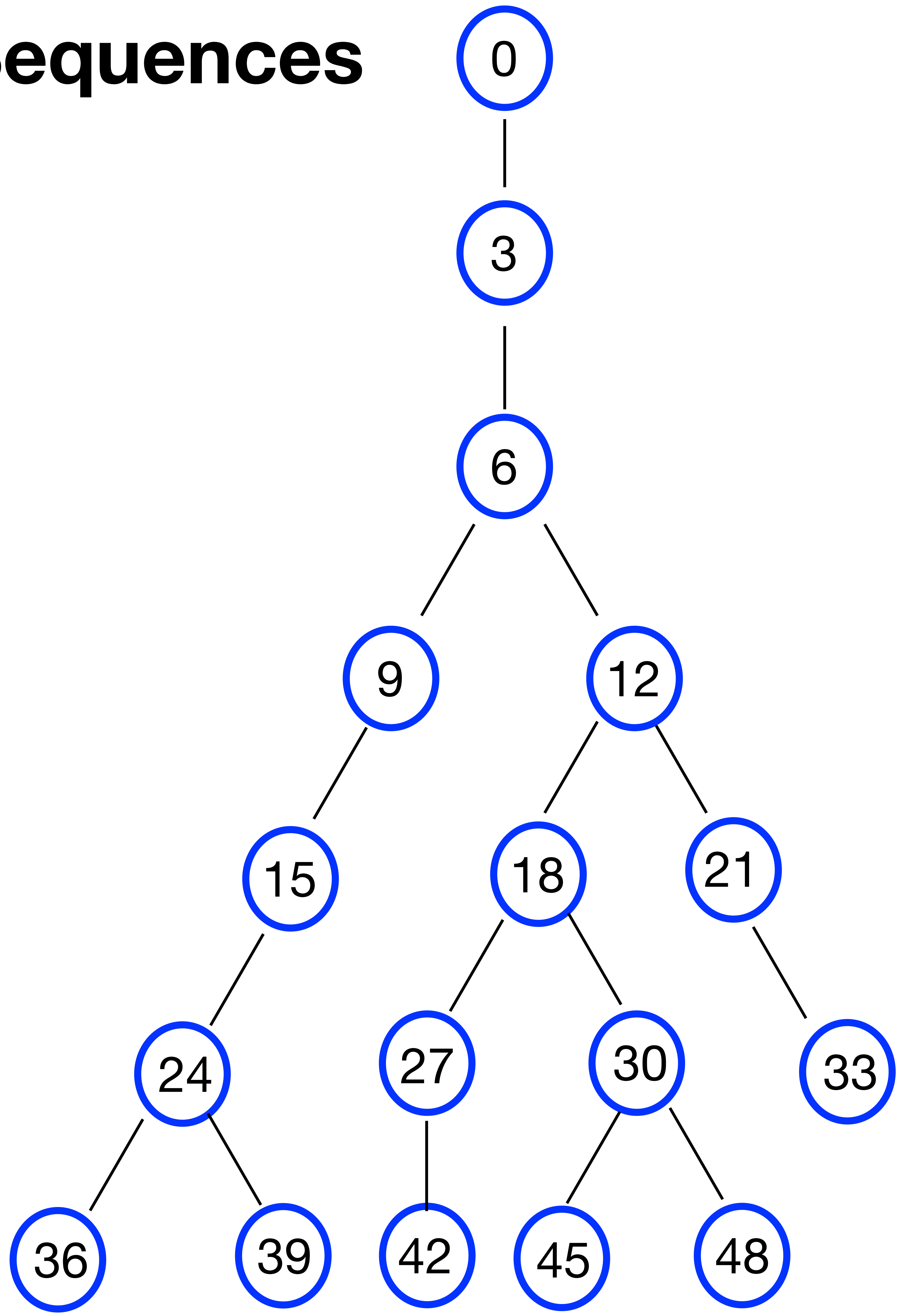

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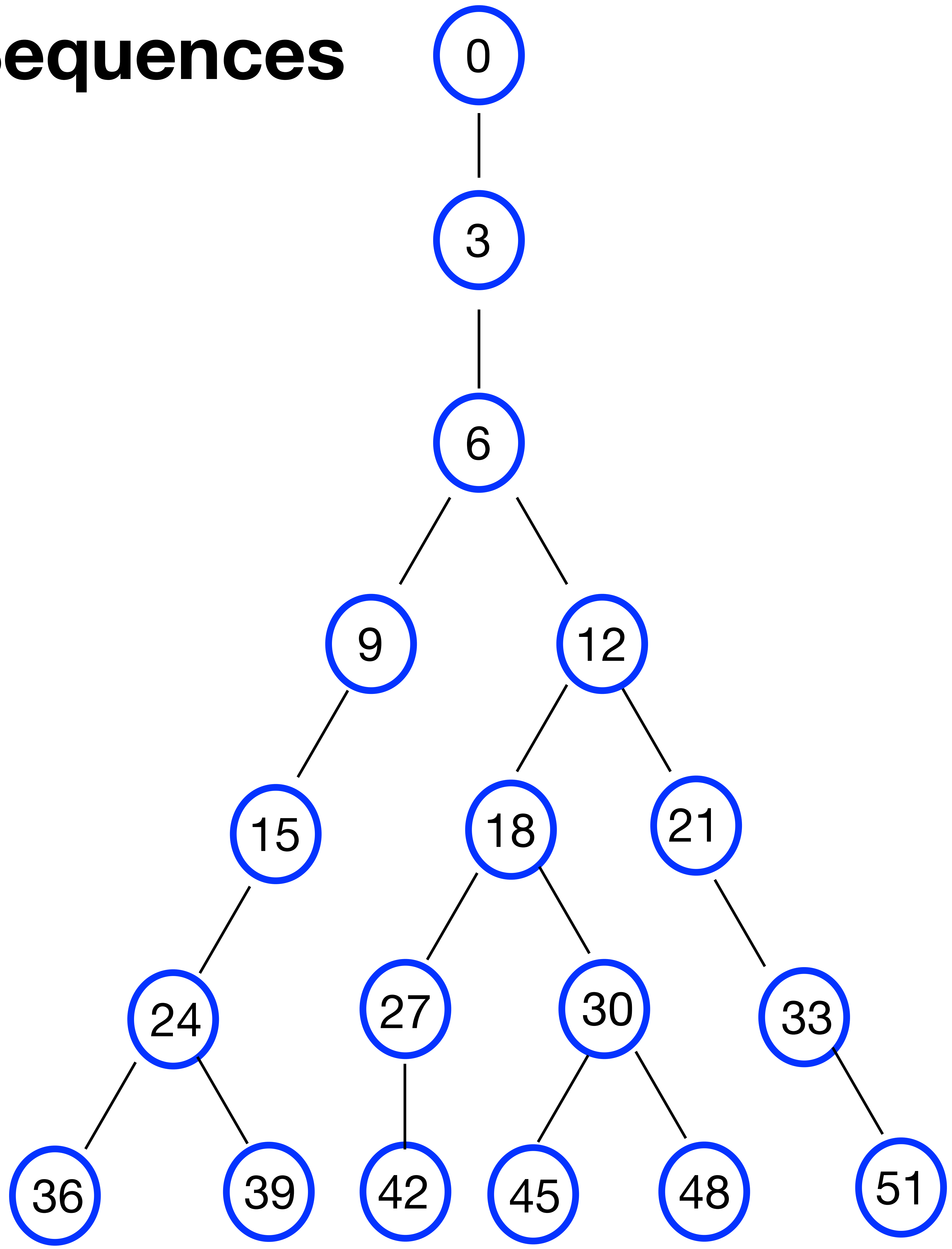

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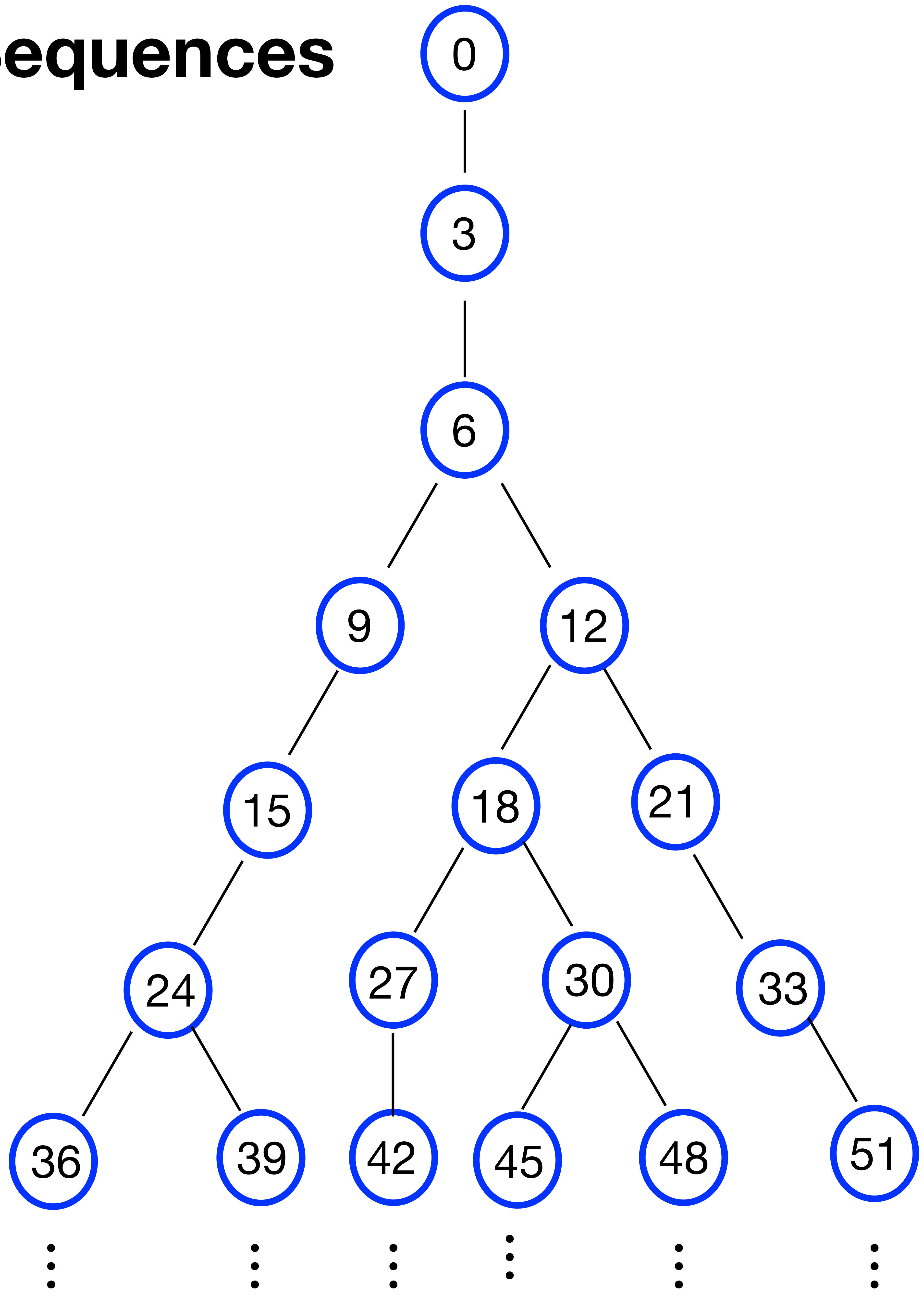

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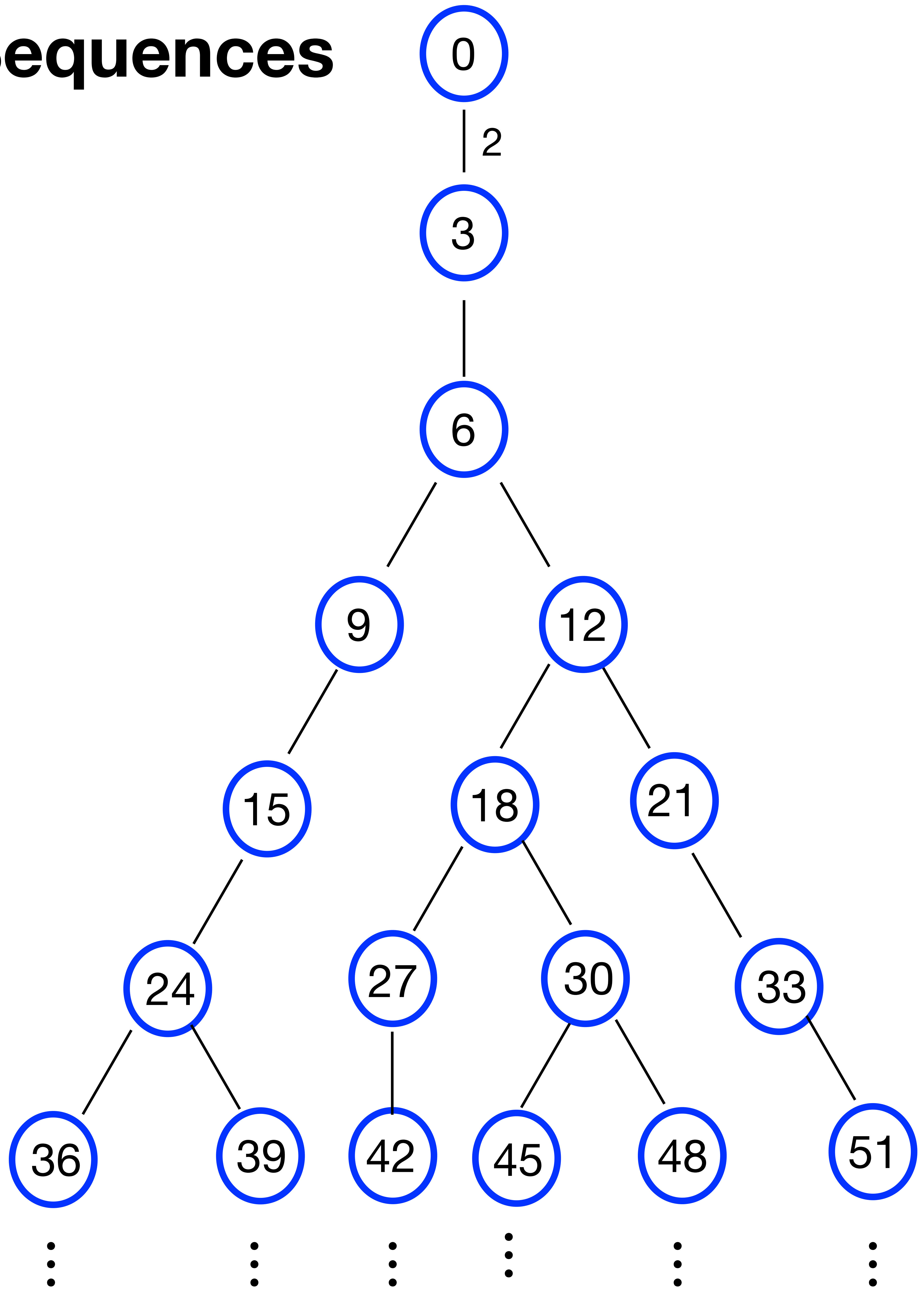

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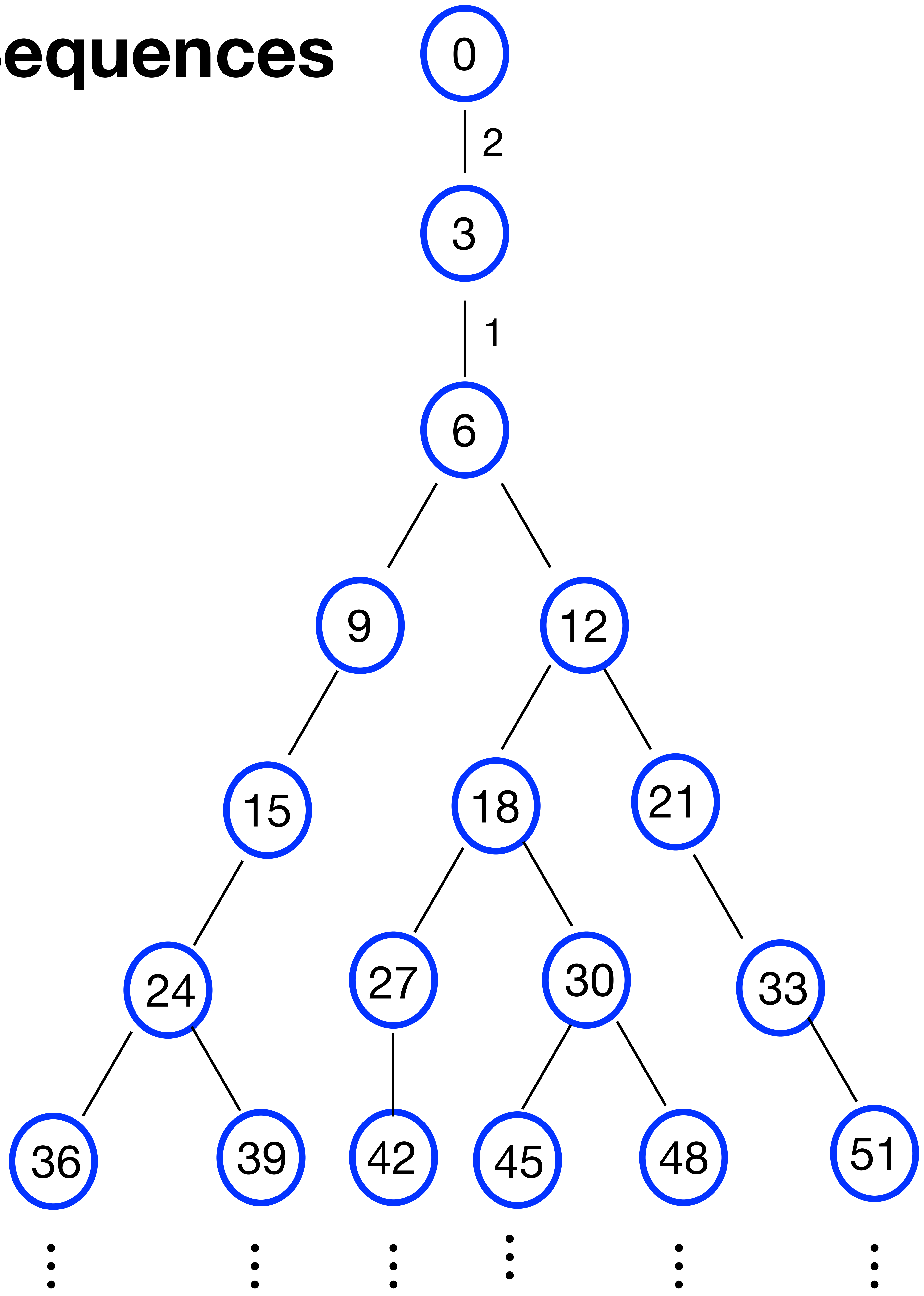

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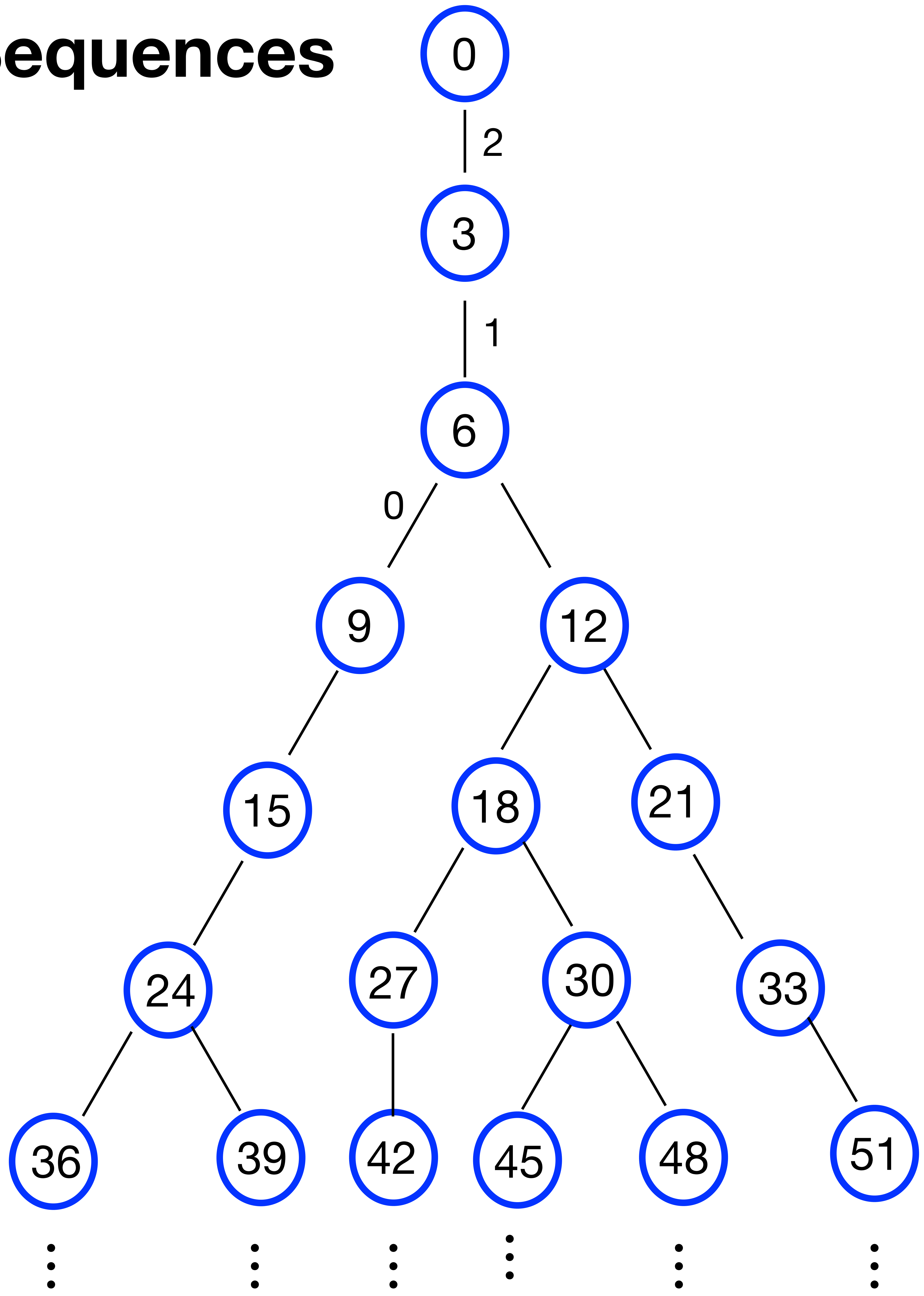

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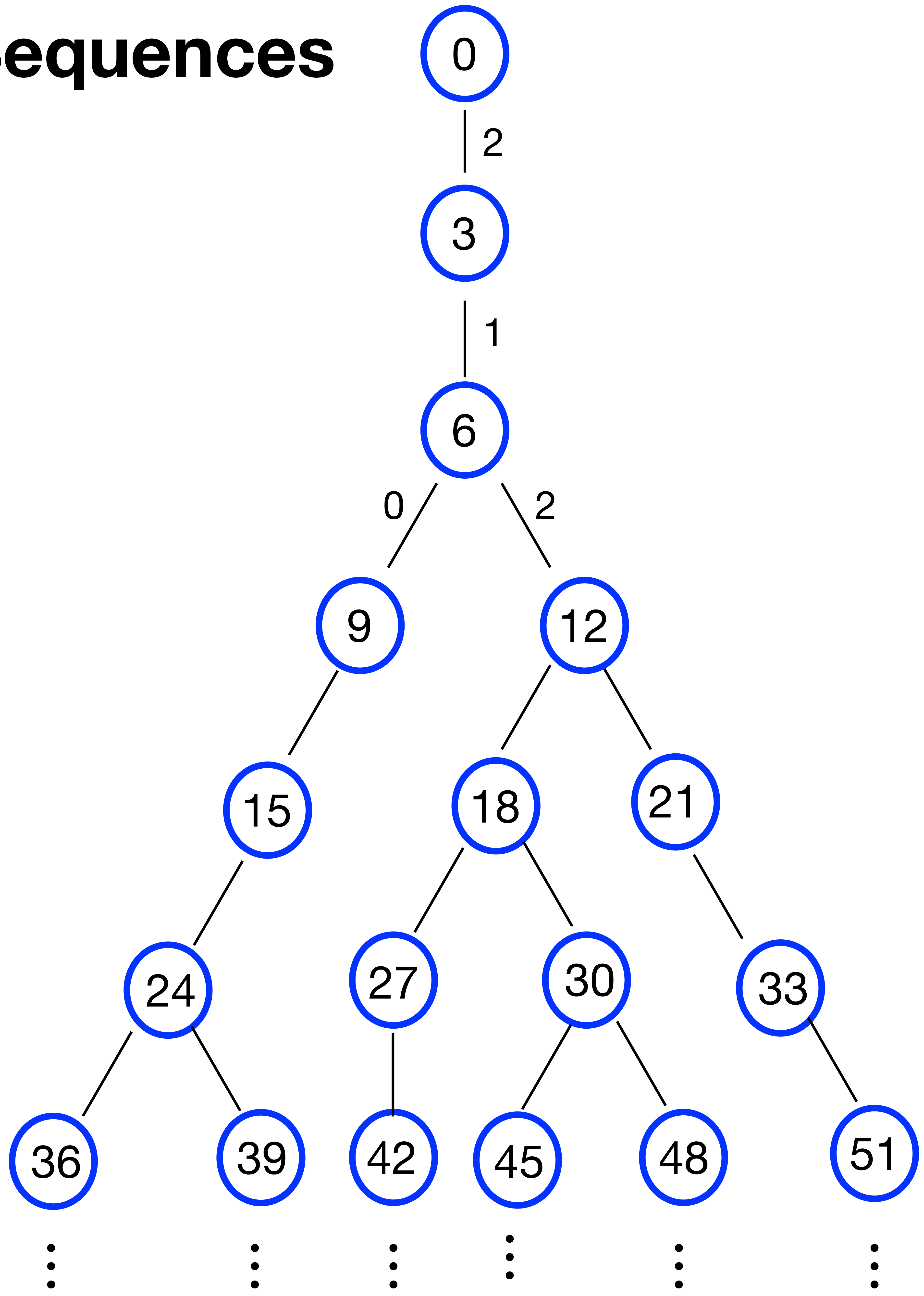

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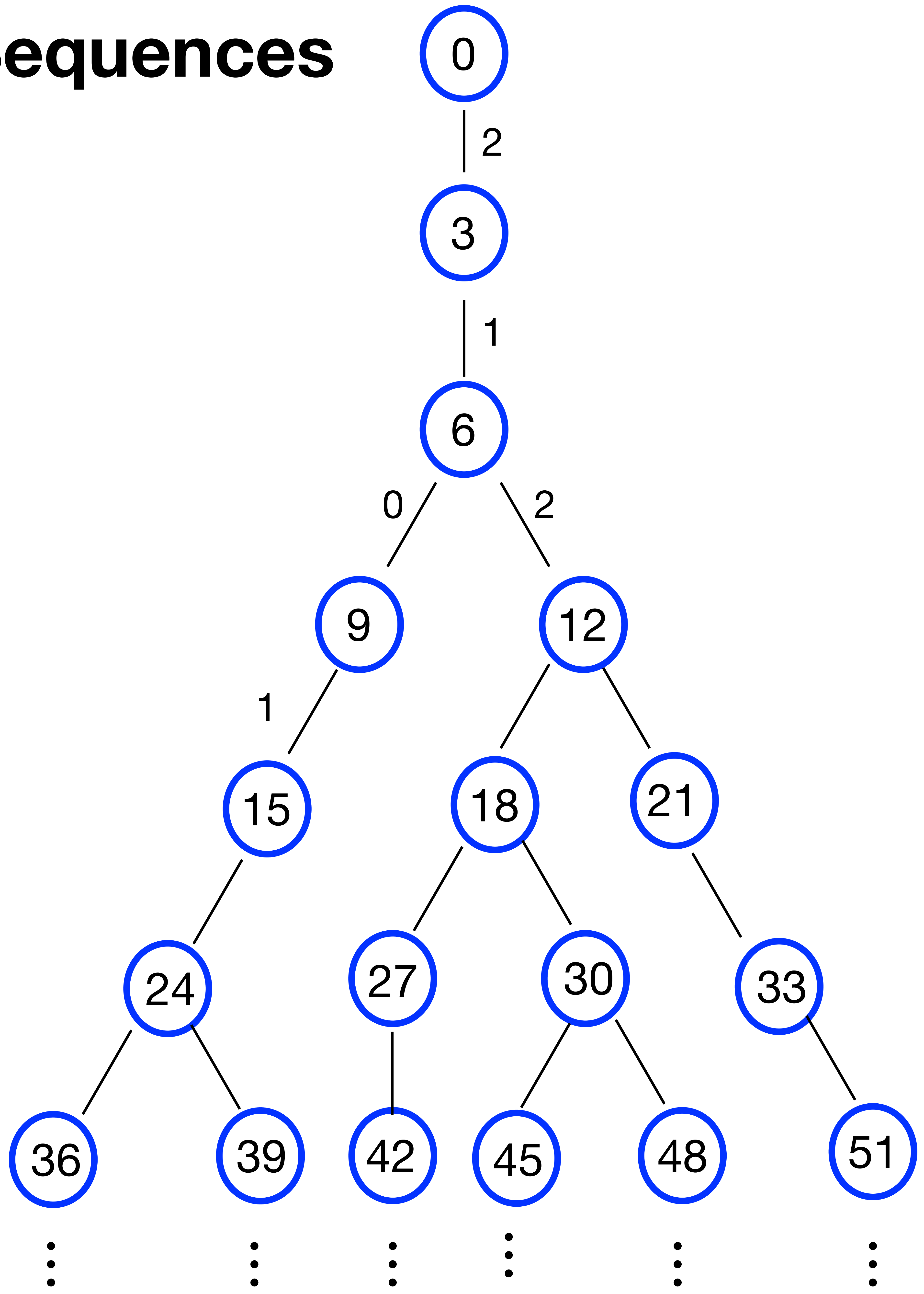

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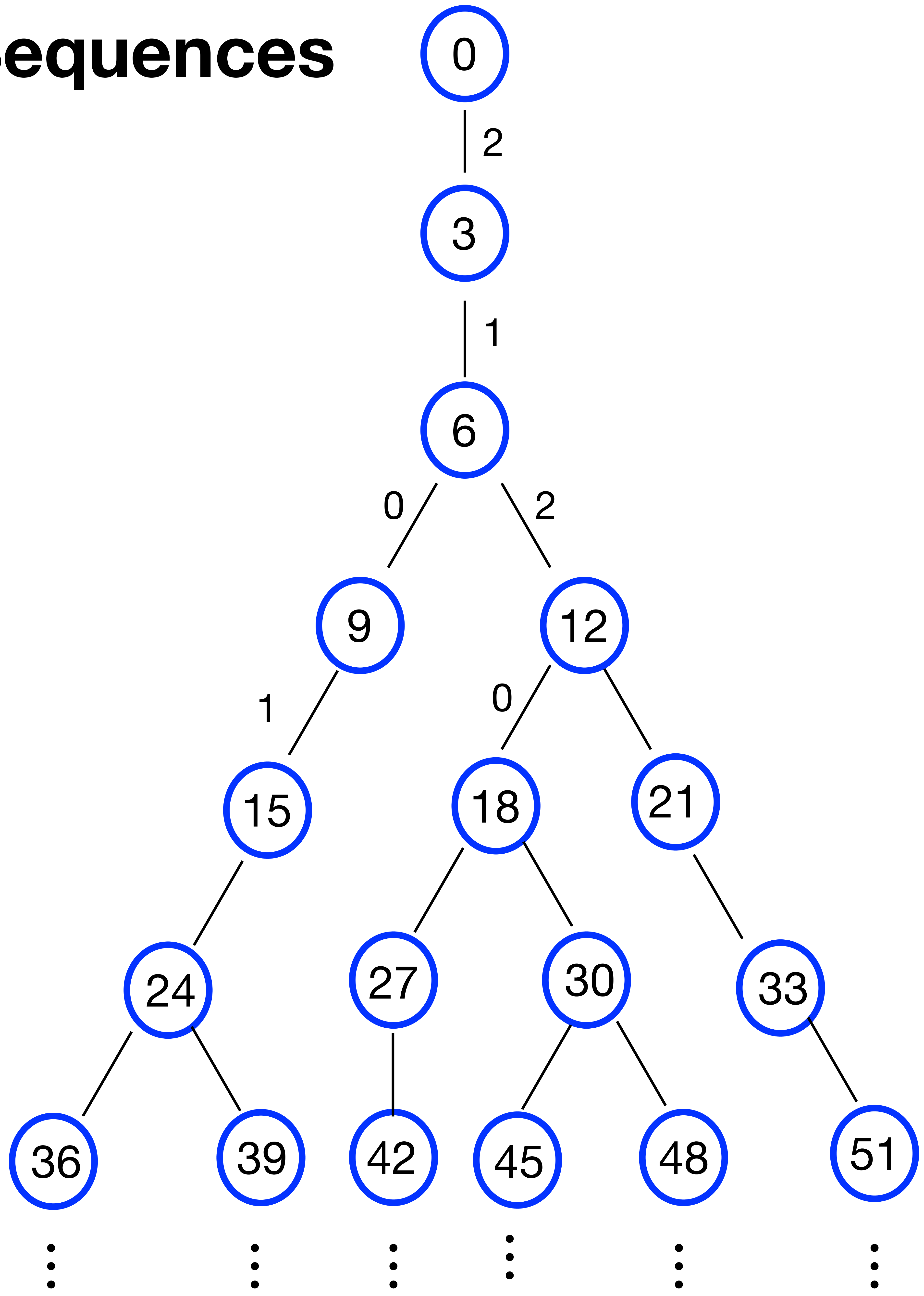

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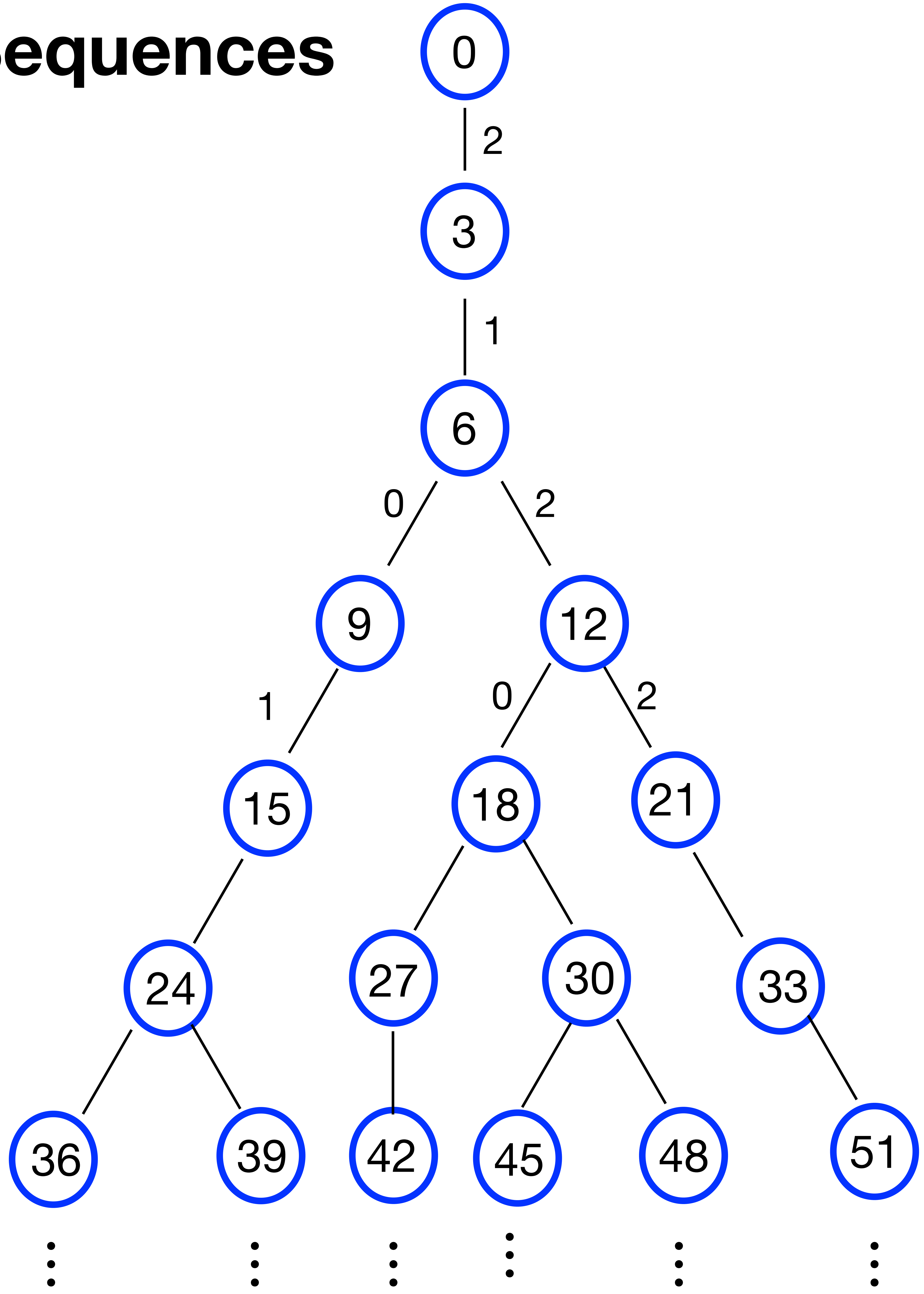

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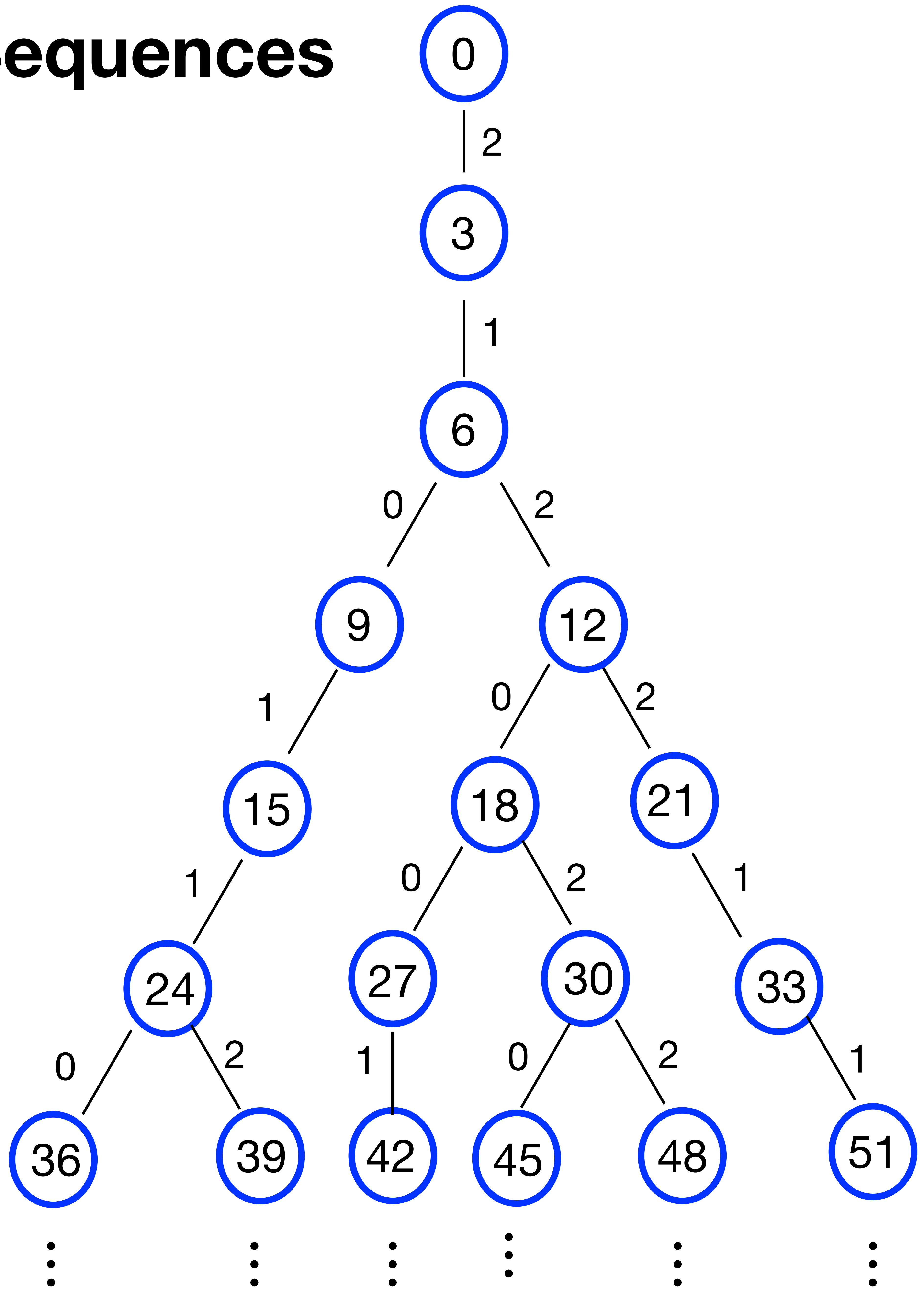
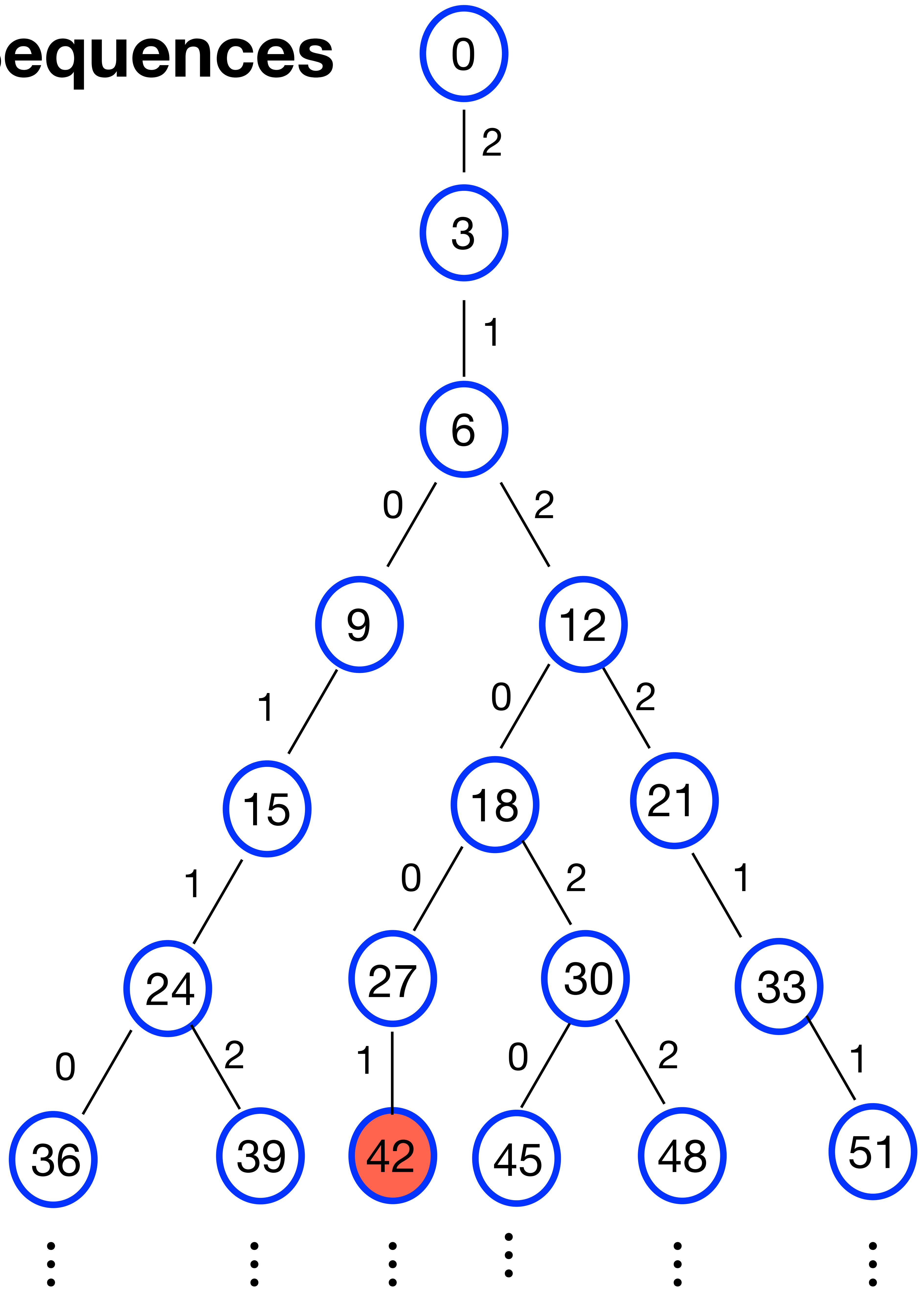


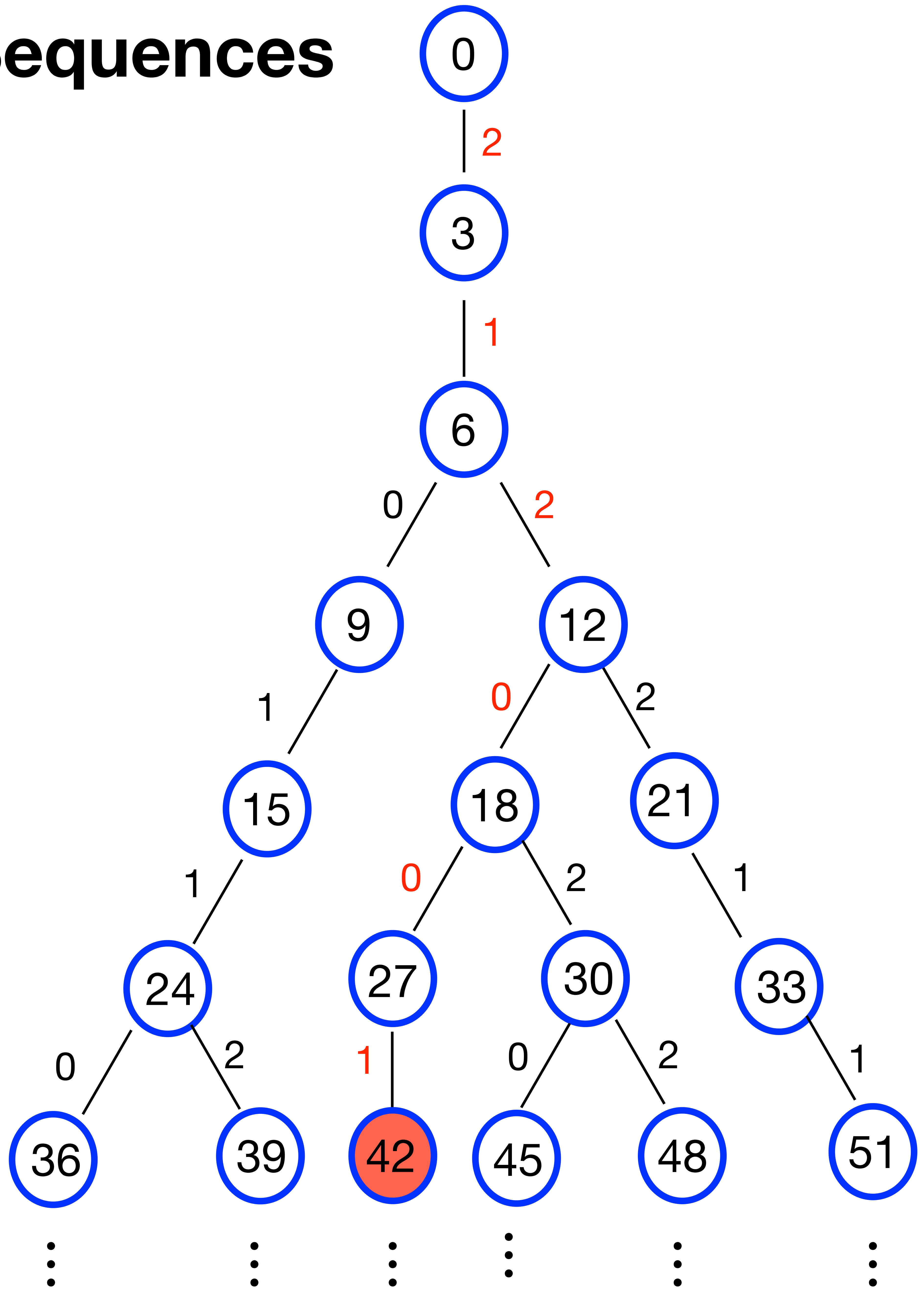
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)



Bonus: Rational Beatty Sequences

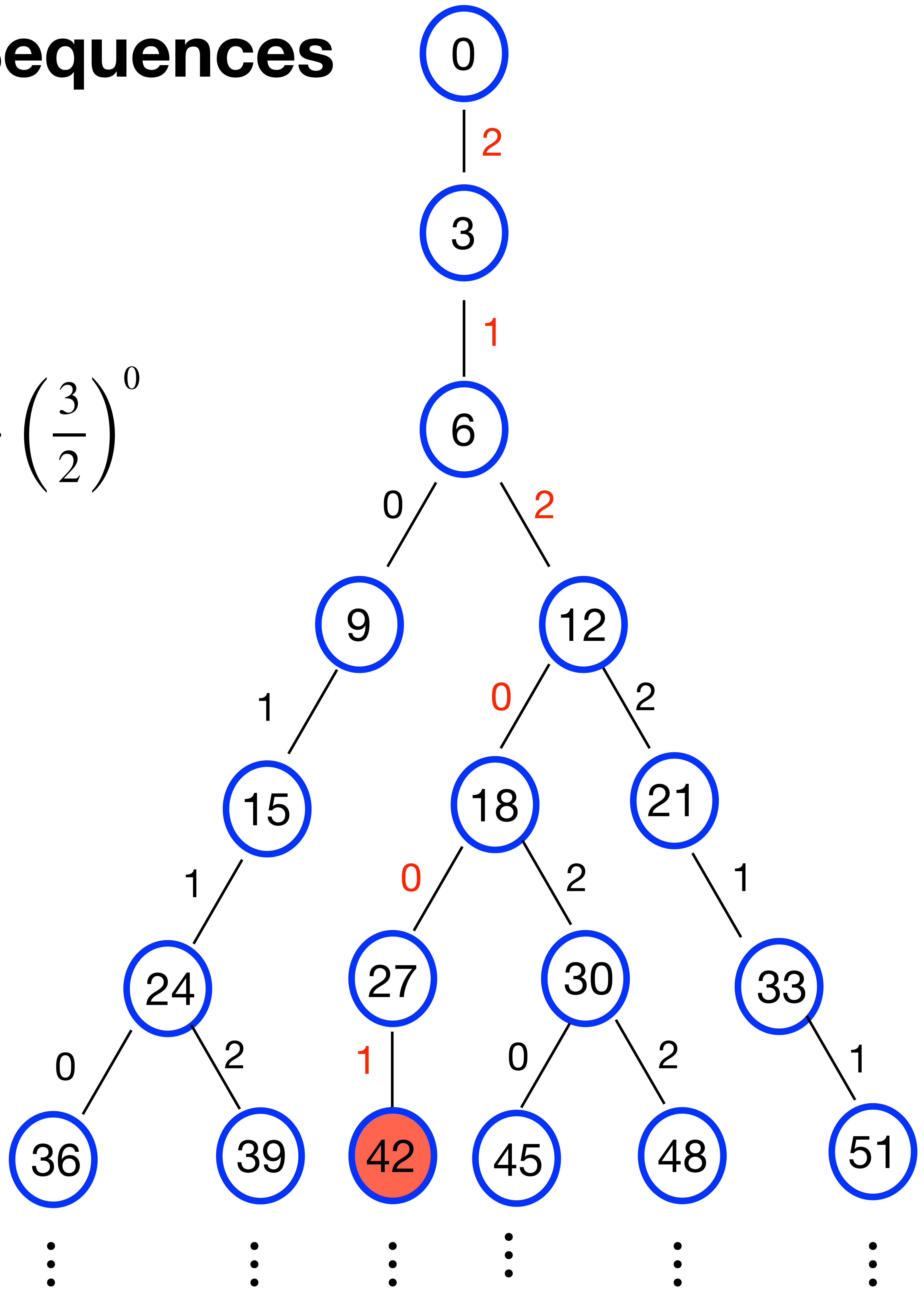
(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)



Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

$$42 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 0 \cdot \left(\frac{3}{2}\right)^0$$

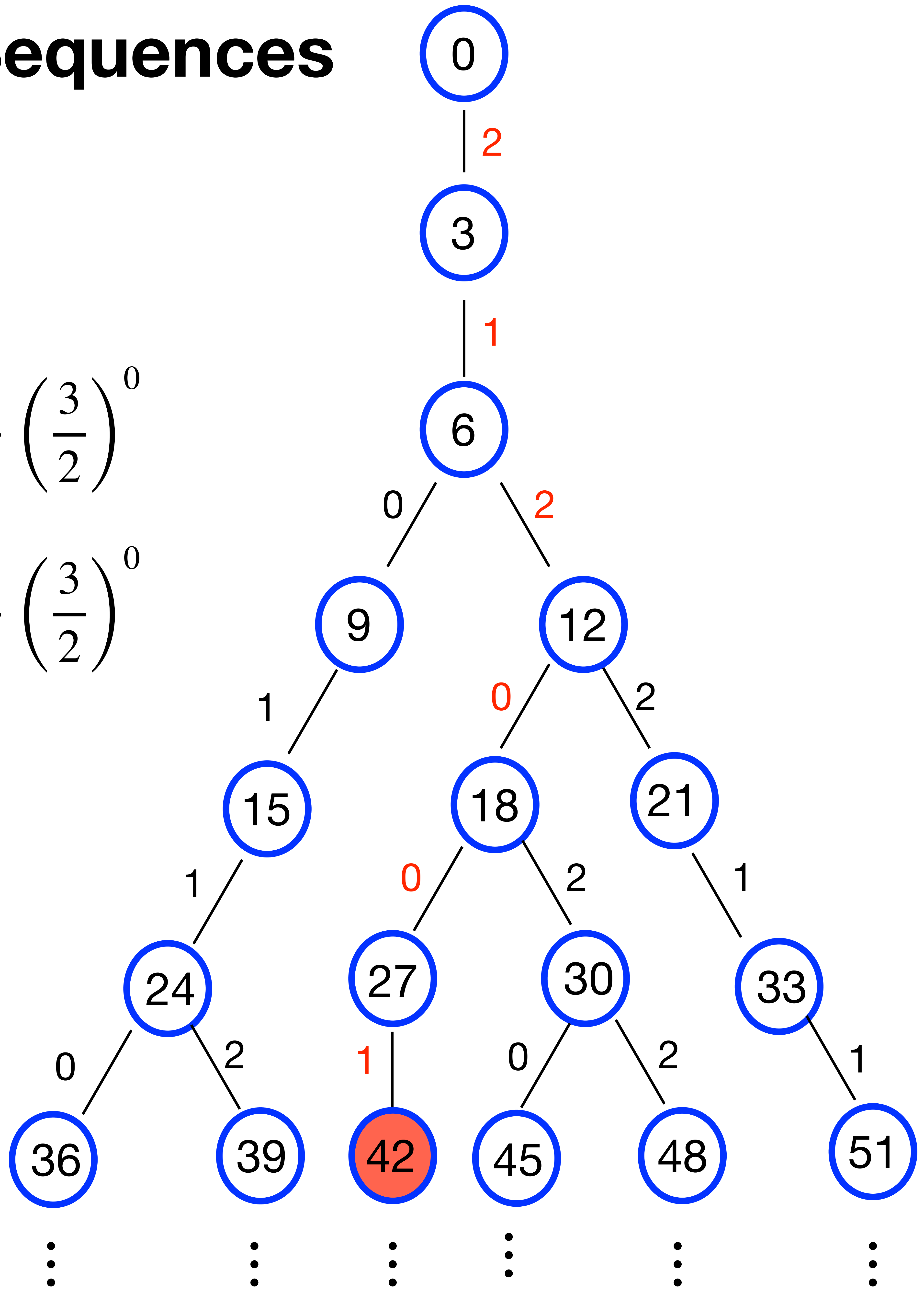


Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

$$42 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 0 \cdot \left(\frac{3}{2}\right)^0$$

$$43 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 1 \cdot \left(\frac{3}{2}\right)^0$$



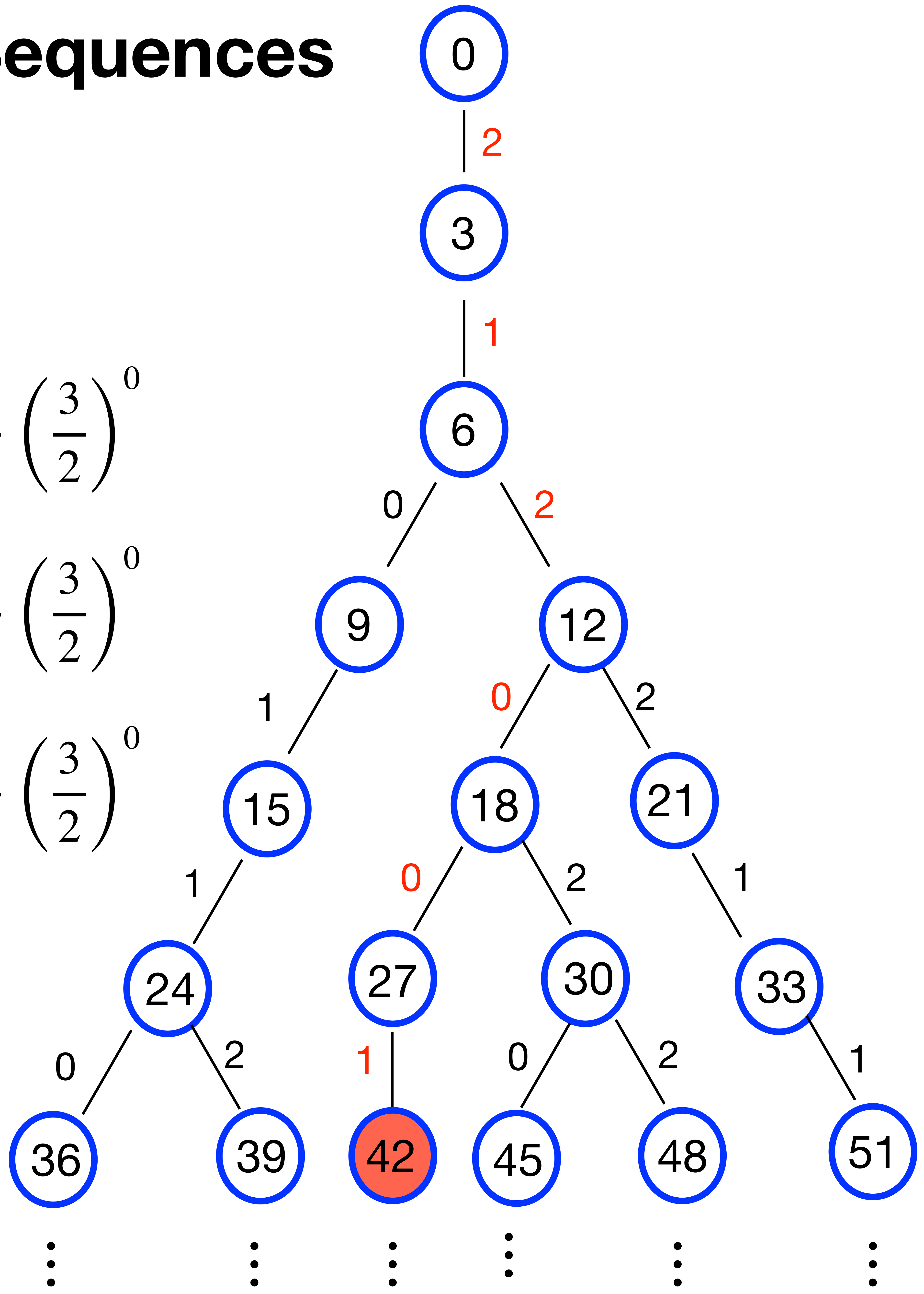
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

$$42 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 0 \cdot \left(\frac{3}{2}\right)^0$$

$$43 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 1 \cdot \left(\frac{3}{2}\right)^0$$

$$44 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 2 \cdot \left(\frac{3}{2}\right)^0$$



Bonus: Rational Beatty Sequences

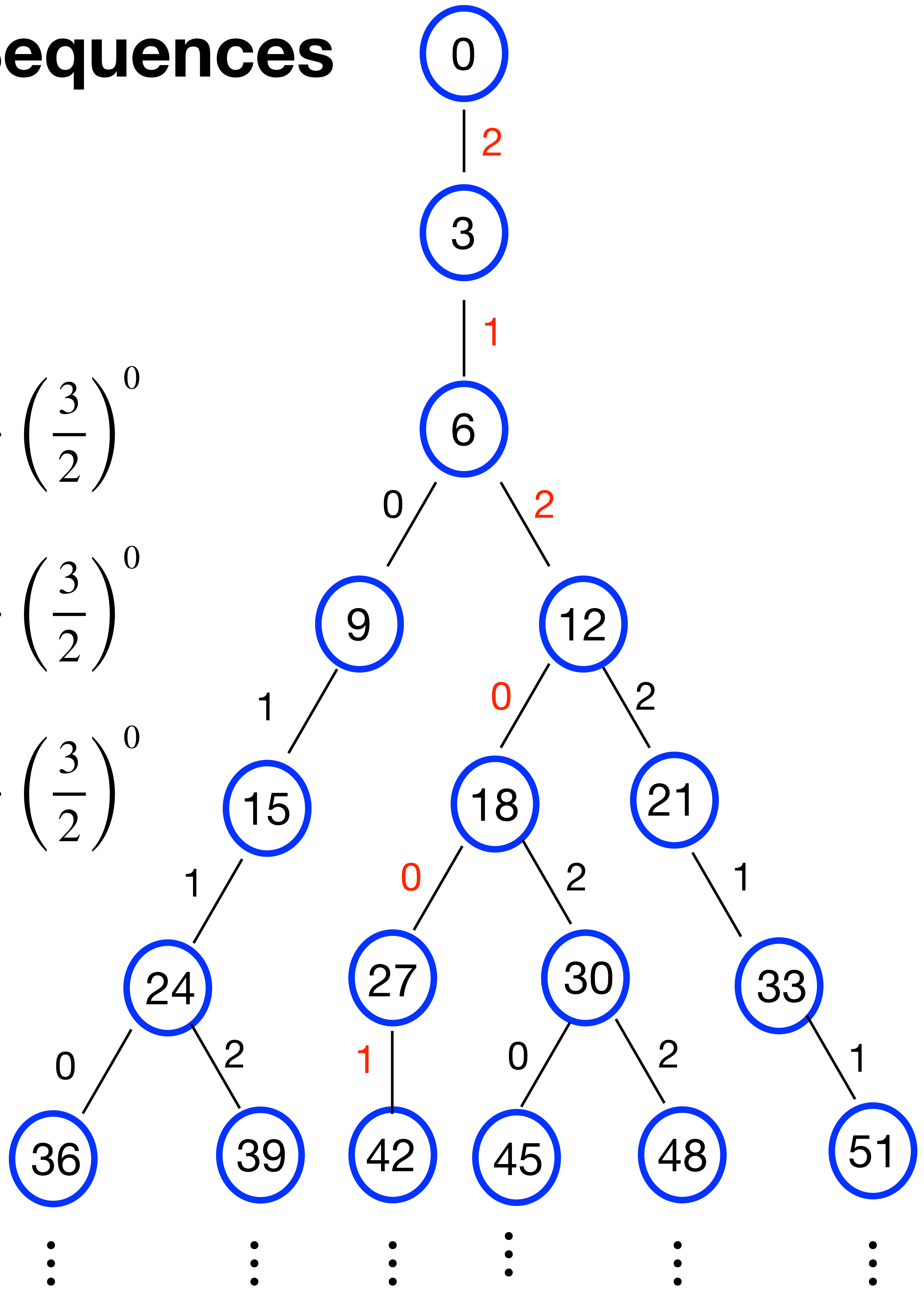
(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

$$42 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 0 \cdot \left(\frac{3}{2}\right)^0$$

$$43 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 1 \cdot \left(\frac{3}{2}\right)^0$$

$$44 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 2 \cdot \left(\frac{3}{2}\right)^0$$

This tree encodes the "base-3/2 representations" of counting numbers - each number is given as a sum of powers of 3/2, with coefficients 0, 1, or 2.



Thanks for listening!

If you have questions, I am happy to answer them.

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