Clopen Sets

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Theorem 1. *The only clopen subsets (both open and closed) of* $\mathbf{R}^n$ *are* $\mathbf{R}^n$ *and* $\emptyset$.

Proof. Suppose A is clopen and not $\mathbf{R}^n$ or $\emptyset$. Then there are points $a \in A$ and $b \notin A$. Let $q(t) = tb + (1-t)a,\ 0 \leq t \leq 1$. Let $t_0 = \sup\{t : q(t) \in A\}$. Since A and A^c are open, $0 < t_0 < 1$. Where is $q(t_0)$? If $q(t_0) \in A$ then t_0 is not the $\sup\{t : q(t) \in A\}$, since A is open. If $q(t_0) \in A^c$ then t_0 is not the $\sup\{t : q(t) \in A\}$, since A^c is open. This contradiction shows that either A or A^c is empty, so either $A = \emptyset$ or $A = \mathbf{R}^n$.

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