Jordan Measurability

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A bounded set E in the plane is *Jordan Measurable* if χ_E is Riemann integrable. χ_E is discontinuous exactly on ∂E, so from a general theorem, we have

Theorem 1. *A bounded set E is Jordan measurable if and only if the Lebesgue measure of ∂E is 0.*

However there is a better theorem:

Theorem 2. *A bounded set E is Jordan measurable if and only if the Jordan measure of ∂E is 0.*

Corollary 1. *The boundary of a bounded set is of Lebesgue measure 0 if and only if it is of Jordan measure 0.*

The corollary can be proved directly using the Heine-Borel theorem.

To prove Theorem 2 we start with a lemma.

Lemma 1. *A set E is of Jordan measure 0 if and only if for every $\epsilon > 0$ there is a finite union of rectangles, $\bigcup_1^n R_i$, with sides parallel the the axis lines, so that $E \subset \bigcup_1^n R_i$ and $\sum_1^n |R_i| < \epsilon$.*

Proof. If E has Jordan measure 0 then the upper sums $S_P(\chi_E)$ can be made as small as we please. This gives a finite set of rectangles satisfying the requirement. On the other had if we have a set of rectangles with $\sum_1^n |R_i| < \epsilon/2$ and $E \subset \bigcup_1^n R_i$, then by fattening them up slightly we can assume they are open. Then taking a partition P that makes all edges of these rectangles unions of rectangles in the partition, we find that we can make $S_P(\chi_E) < \epsilon$.

□

Proof. (of Theorem 2.) Suppose E is Jordan measurable. Then there is a partition P such that $\partial E \subset \bigcup \widetilde{R}_{ij}$, where $\widetilde{R}_{ij}$ are special rectangles and $\sum |\widetilde{R}_{ij}| = S_P(\chi_E) - s_P(\chi_E) < \epsilon$.

For the reverse direction, suppose $|\partial E| = 0$. Then choose open rectangles such that $\partial E \subset \bigcup_1^n R_i$ and $\sum_1^n |R_i| < \epsilon$. Now choose a partition P so that these rectangles are unions of rectangles defined by the partition. Then every rectangle not included in this union either consists entirely of points of E or entirely of points of E^c. Hence every special rectangle (see definition of special rectangles in the remark following) for P and E is included in this union. Thus $\sum |R_{ij}| = S_P(\chi_E) - s_P(\chi_E) < \epsilon$ and χ_E is integrable.

□

REMARK. Here's another argument. Let P be a partition and let $\widetilde{R}_{ij}$ be the special rectangles for E in this partition. Recall the *special rectangles* are characterized by the property that $\widetilde{R}_{ij} \cup E \neq \emptyset$ and $\widetilde{R}_{ij} \cup E^c \neq \emptyset$. By looking at separate cases, it's not too hard to see that $\partial E \subset \bigcup \widetilde{R}_{ij}$. Here's a summary of that argument. If $p \in \partial E$ is in the interior of R_{ij}, then $R_{ij} \cup E \neq \emptyset$ and $R_{ij} \cup E^c \neq \emptyset$. If $p \in \partial E$ is on the boundary of some rectangle, then: if $p \notin E$ then there is a point in one of the neighboring rectangles that is in E; if $p \in E$, then there is a point in a neighboring rectangle that is not in E. So in every case, if $p \in E$, then $p \in \widetilde{R}_{ij}$ for some special rectangle $\widetilde{R}_{ij}$.

We now have (for any partition, P),

$$S_P(\chi_E) - s_P(\chi_E) = \sum |\widetilde{R}_{ij}|. \tag{1}$$

Taking inf's,

$$\overline{A}(E) - \underline{A}(E) = \inf_P \{\sum |\widetilde{R}_{ij}|\} \tag{2}$$

Since $\partial E \subset \bigcup \widetilde{R}_{ij}$,

$$\overline{A}(\partial E) \leq \overline{A}\left(\bigcup \widetilde{R}_{ij}\right) = \sum |\widetilde{R}_{ij}|. \tag{3}$$

Now take inf's to get

$$\overline{A}(\partial E) \leq \overline{A}(E) - \underline{A}(E) \tag{4}$$

Now take any special rectangle. Since it contains a point in E and a point in E^c and since it is convex it contains the line segment joining these two points. One of the points on this line segment must be a point of ∂E. Hence every special rectangle contains a point of ∂E. That means that every special rectangle contributes to the upper sum for ∂E. In other words,

$$S_P(\chi_{\partial E}) \geq \sum |\widetilde{R}_{ij}|. \tag{5}$$

Take inf's of both sides to get

$$\overline{A}(\partial E) \geq \inf_P \{\sum |\widetilde{R}_{ij}|\} = \overline{A}(E) - \underline{A}(E) \tag{6}$$

and we get

$$\overline{A}(E) - \underline{A}(E) = \overline{A}(\partial E),$$

whether E is measurable or not. In particular E is measurable if and only if $\overline{A}(\partial E) = 0$.