Area of the n-sphere

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This note will derive the following result.

Theorem 1. *Let* $S_n = \{(x_1, x_2, \ldots, x_{n+1}) : |x|^2 = \sum_1^{n+1} x_j^2 = a^2\}$ *be the n-sphere of radius a. Denote the area of this sphere by* $A_n(a)$*. It satisfies thefollowing recursion:*

$$
\begin{aligned}
A_n(a) &= \alpha_n a^n, \ \textit{where} \\
\alpha_n &= \left(\frac{2\pi}{n-1}\right)\alpha_{n-2} \\
\alpha_0 &= 2 \\
\alpha_1 &= 2\pi
\end{aligned}
$$

Proof. We know $A_0(a) = 2, A_1(a) = 2\pi a$. The area formula is

$$
\begin{aligned}
A_n(a) &= 2\int_{x_1^2+\ldots x_n^2 \le a^2} \left[1 + \frac{x_1^2 + \ldots x_n^2}{a^2 - x_1^2 - \cdots - x_n^2}\right]^{1/2} dx_1 \ldots dx_n \\
&= 2\int_{x_1^2+x_2^2 \le a^2} \left(\int_{x_3^2+\cdots+x_n^2 \le a^2-x_1^2-x_2^2} \left[\frac{a^2}{a^2 - x_1^2 - \cdots - x_n^2}\right]^{1/2} dx_3 \ldots dx_n\right) dx_1 dx_2 \\
&= 2a\int_{x_1^2+x_2^2 \le a^2} \left(\int_{x_3^2+\cdots+x_n^2 \le a^2-x_1^2-x_2^2} \left[\frac{1}{a^2 - (x_1^2+x_2^2)}\right]^{1/2} \left[\frac{a^2 - (x_1^2+x_2^2)}{a^2 - x_1^2 - \cdots - x_n^2}\right]^{1/2} dx_3 \ldots dx_n\right) dx_1 dx_2 \\
&= 2a\int_0^{2\pi}\int_0^a \left[\frac{1}{a^2-r^2}\right]^{1/2} \left(\int_{x_3^2+\cdots+x_n^2 \le a^2-r^2} \left[\frac{a^2-r^2}{a^2-r^2-x_3^2-\cdots-x_n^2}\right]^{1/2} dx_3 \ldots dx_n\right) r dr d\theta \\
&= 2a\int_0^{2\pi}\int_0^a \left[\frac{1}{a^2-r^2}\right]^{1/2} \frac{A_{n-2}(\sqrt{a^2-r^2})}{2} r dr d\theta \\
&= 2\pi a \alpha_{n-2}\int_0^a (a^2-r^2)^{(n-3)/2} r dr \\
&= \left(\frac{2\pi\alpha_{n-2}}{n-1}\right) a^n
\end{aligned}
$$

□