9/25/08

Cauchy – Schwartz

$$\sum a_i^2 \cdot \sum b_i^2 \geq \left(\sum a_i b_i\right)^2$$

actually

$$(*)\quad \begin{cases} \sum a_i^2 \cdot \sum b_i^2 - \left(\sum a_i b_i\right)^2 \\ = \sum_{i<j} (a_i b_j - a_j b_i)^2 \\ = \frac{1}{2} \sum_{i,j} (a_i b_j - a_j b_i)^2 = \frac{1}{2} \sum \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}^2 \end{cases}$$

pf: easy to see that second two are same

$$\sum_{i,j} (a_i b_j - a_j b_i)^2 = \sum_{i,j} \left[a_i^2 b_j^2 + a_j^2 b_i^2 - 2 a_i b_i a_j b_j \right]$$

$$= 2 \sum a_i^2 \sum b_j^2 - 2 \left(\sum a_i b_i\right)^2 \qquad \text{QED.}$$

Interpretation: cross product $A \times B$

$A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$

$$A \times B = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

$$|A \times B|^2 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$$

$$= |A|^2 |B|^2 - (A \cdot B)^2$$

$$= |A|^2 |B|^2 (1 - \cos^2\theta) = |A|^2 |B|^2 \sin^2\theta$$

$\therefore$ $|A \times B|$ = area of ‖-ogram spanned by A & B.

the identity (*) generalizes this:

$A, B \in \mathbb{R}^n$, area of

$\|$-ogram spanned by A & B is

$$\sqrt{\sum_{i<j} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}^2} = |A| \cdot |B| \sin\theta$$

(take all 2×2 dets formed from the $2 \times n$ matrix $\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix}$)