Manifolds

Note Title 11/5/2008

Curves and surfaces are special cases of mathematical objects called manifolds. As geometric objects curves and surfaces are often thought of as sitting inside a larger manifold, frequently $\mathbb{R}^2, \mathbb{R}^3, \cdots, \mathbb{R}^n$ and as such are submanifolds.

What I defined on Wednesday, Nov. 5, was the concept of 1-dimensional submanifold of $\mathbb{R}^2$. For some discussions a curve should be connected, so a "curve in $\mathbb{R}^2$" would be a connected 1-dimensional submanifold of $\mathbb{R}^2$. Similarly we might limit the use of the phrase "surface in $\mathbb{R}^3$" to mean a 2-dimensional submanifold of $\mathbb{R}^3$. If we want these objects to have well-defined tangent spaces we would (and should) call the previous objects "topological submanifolds" and the ones with tangent spaces "differentiable submanifolds".

They may or may not be connected.

Here is the precise definition.

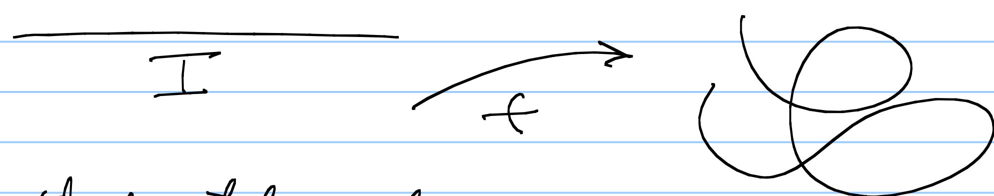
Let $M \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. M is a "submanifold of $\mathbb{R}^n$ of dimension m" if for every point $p \in M$, there is an open set W, $p \in W \subset \mathbb{R}^n$ and 1-1 continuous function ("map") $\Phi : W \to U \subset \mathbb{R}^n$ onto $U = \{y \in \mathbb{R}^n : |y_j| < \epsilon,\ j = 1, \cdots, n\}$ so that $\Phi^{-1} : U \to W$ is continuous and $\Phi(M \cap W) = \{y \in U : y_{m+1} = 0, y_{m+2} = 0, \cdots, y_n = 0\}$.

We say M is of "codimension $n-m$". This is the definition of "<u>topological</u>" submanifold.

For the definition of "<u>differentiable</u>" submanifold change the word continuous in the above definition to differentiable everywhere you see it.

There is an entirely different concept, "<u>parametrized curve</u>" and that is the definition of curve used to define path (or arc) connected. A <u>parametrized curve</u> is a function $f : I \to \mathbb{R}^n$ where I

is an open interval in $\mathbb{R}^1$. So a parametrized curve is <u>not</u> a subset of $\mathbb{R}^n$ and hence shouldn't be called a curve, but that doesn't stop people from calling it a curve. Often what's meant is the image $f(I)$, but that may fail to be a 1-dimensional submanifold of $\mathbb{R}^n$.



Here are the three statements I proved in class

A) Let f be continuously differentiable on the open set $I \subset \mathbb{R}^1$. Then $C = \{(x, f(x)) : x \in I\}$ is a 1-dimensional differentiable submanifold of $\mathbb{R}^2$ (it is also connected).

Proof: Let $x_0 \in I$. Consider the map F
$F(x,y) = (x - x_0, y - f(x))$ on a neighborhood of $(x_0, f(x_0)) \in \mathbb{R}^2$. Notice $(x_0, f(x_0))$ is a typical point of C. $|DF| = \begin{vmatrix} 1 & 0 \\ -f'(x) & 1 \end{vmatrix} = 1 \neq 0$. So by the

inverse function theorem there is a neighborhood W of $(x_0, f(x_0))$ and neighborhood $\{|u|<\epsilon, |v|<\epsilon\} = U$ in $\mathbb{R}^2$ so that $F: W \to U$ is 1-1 and has a differentiable inverse. $F^{-1}(\{(u,v)\in U : v=0\}$

$$= \{(x,y)\in W : u = x-x_0,\ y-f(x) = v = 0\}$$
$$= \{|x-x_0| < \epsilon,\ y = f(x)\} = C \cap W$$

B) Let f be continuously differentiable on an open set V of $\mathbb{R}^2$. Let $C = \{(x,y) : f(x,y)=0,\ (x,y)\in W\}$ Suppose $\nabla f(x,y) \neq 0$ for all points $(x,y)\in C$. Then C is a 1-dimensional submanifold of $\mathbb{R}^2$.

<u>Proof</u>: Let $(x_0, y_0) \in C$ and suppose $f_y(x_0,y_0) \neq 0$.

Let $F(x,y) = (x-x_0, f(x,y))$.

Then $|DF| = \begin{vmatrix} 1 & 0 \\ f_x & f_y \end{vmatrix} = f_y$, so $(DF)(x_0,y_0) \neq 0$.

again there is an open set W, $(x_0,y_0)\in W$, and $U = \{|u|<\epsilon, |v|<\epsilon\}$ with $F: W \to U$ and $F^{-1}: U \to W$ differentiable. as in A)

$$F^{-1}(\{|u|<\epsilon,\ v=0\}) = \{(x,y) : |x-x_0|<\epsilon,\ F(x,y)=0\}$$

$= C \cap W.$

c) Let $\gamma(t) = (\alpha(t), \beta(t))$ be a continuously differentiable parametrized curve with $\gamma'(t) \neq (0,0)$, (where γ is defined on some open interval $I \subset \mathbb{R}^1$). Let $t_0 \in I$. Then there is an $\epsilon > 0$ so that

$\{(\alpha(t), \beta(t)) : |t - t_0| < \epsilon\}$ is a 1-dimensional manifold. (It's also connected.)

Proof: Let $\alpha'(t_0) \neq 0$. Let $F(t,v) = (\alpha(t+t_0), v + \beta(t+t_0))$

Then $|DF| = \begin{vmatrix} \alpha'(t_0) & 0 \\ \beta'(t_0) & 1 \end{vmatrix} = \alpha'(t_0) \neq 0$, at $t=0, v=0$.

$$F(t,0) = (\alpha(t+t_0), \beta(t+t_0))$$

There is an open set W with $(\alpha(t_0), \beta(t_0)) \in W$ and an $\epsilon > 0$ so that $U = \{|t| < \epsilon, |v| < \epsilon\}$ maps 1-1 differentiably by F onto W and F^{-1} is differentiable.

$\{(t,v) : v = 0, |t| < \epsilon\} \subset U$

$\overset{F}{\longleftrightarrow} \{(\alpha(t+t_0), \beta(t+t_0)) : |t| < \epsilon\} \subset W.$

$= \{(\alpha(t), \beta(t)) : |t - t_0| < \epsilon\}.$

So $\{(\alpha(t), \beta(t)) ; |t - t_0| < \epsilon$ is a 1-dim. manifold.