Partial Derivative Relation

Note Title 10/28/2008

Some of you have asked about appropriate assumptions on the partial derivatives of F in problem #6, §2.5. Do we need to assume $F_z \neq 0$, etc.? The answer is no, we don't need an assumption; we don't even need F. Here is the general result

<u>Theorem</u>: Suppose a set S in $\mathbb{R}^n$ can be described in n different ways: as a graph $x_1 = f_1(x_2, \ldots, x_n)$; as a graph $x_2 = f_2(x_1, \hat{x}_2, \ldots, x_n)$; as a graph $x_3 = f_3(x_1, x_2, \hat{x}_3, \ldots, x_n)$; --- ; as a graph $x_n = f_n(x_1, \ldots, x_{n-1})$; where $\hat{x}_j$ means omit variable x_j. Then

$$(f_1)_{x_2} \cdot (f_2)_{x_3} \cdot \cdots \cdot (f_n)_{x_1} = (-1)^n.$$

<u>Proof</u>: The assumption implies that S has a well defined tangent plane Π at each point. Let's assume $0 \in S$ (out of convenience). Then the tangent plane goes through the origin and has an equation of the form $A_1x_1 + A_2x_2 + \cdots + A_nx_n = 0$.

$(A_1, A_2, \dots, A_n)$ is normal to the plane. The normal is uniquely determined up to a non-zero multiple (all normals are proportional). Because S is the graph of $f_1(x_2, \dots, x_n)$, $(1, -(f_1)_{x_2}, \dots, -(f_1)_{x_n})$ is normal to Π, $-(f_1)_{x_2} = \frac{A_2}{A_1}$. Similarly,

$$-(f_2)_{x_3} = \frac{A_3}{A_2}, \dots, -(f_n)_{x_1} = \frac{A_1}{A_n}$$

Now multiply all of these to get

$$1 = \frac{A_2}{A_1} \cdot \frac{A_3}{A_2} \cdot \dots, \frac{A_1}{A_n} = (-1)^n (f_1)_{x_2} (f_2)_{x_3} \cdots (f_n)_{x_1}$$

(the homework problem was $\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1 = (-1)^3$.)