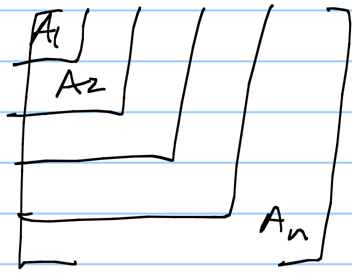
Positive Definiteness

Note Title 10/31/2008

<u>Theorem</u>: Let A be a symmetric $n \times n$ matrix. Let $A_1, A_2, \dots, A_n$ be the principal submatrices.



Then A is positive definite if and only if $\det A_1 > 0$, $\det A_2 > 0$, $\dots$, $\det A_n > 0$.

Before I start the proof I'll explain block multiplication of matrices. Let T and S be two matrices partitioned as follows.

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = T, \quad \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = S$$

A_1 is $m_1 \times n_1$, B_1 is $m_1 \times n_2$, C_1 is $m_2 \times n_1$, D_1 is $m_2 \times n_2$

A_2 is $n_1 \times p_1$, B_2 is $n_1 \times p_2$, C_2 is $n_2 \times p_1$, D_2 is $n_2 \times p_2$

Then

$$T \cdot S = \begin{bmatrix} A_1 A_2 + B_1 C_2 & A_1 B_2 + B_1 D_2 \\ C_1 A_2 + D_1 C_2 & C_1 B_2 + D_1 D_2 \end{bmatrix}$$

We say that T and S are <u>partitioned</u>.

We can also partition vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

so that $Tx = \begin{bmatrix} A_1 x_1 + B_1 x_2 \\ C_1 x_1 + D_1 x_2 \end{bmatrix}$.

Now we begin the proof

I. Suppose A is positive definite. Let J be a multi-index $J = (j_1, \ldots, j_k)$

$1 \leq j_1 < j_2 < \cdots < j_k \leq n$, $k \leq n$. Let $A(J,J)$ be the submatrix of A consisting of rows J and columns J. The (r,s) entry of $A(J,J)$ is $a_{j_r j_s}$.

Let x be any k-vector and let X be the n-vector with

$$X_j = \begin{cases} x_p & \text{if } j = j_p, \text{ for some } p, \\ 0 & \text{if } j \neq j_p, \text{ for any } p. \end{cases}$$

In other words put x in the J locations and set the other values of X_j to be 0. Then

$X^T A X = x^T A(J,J) x \geqslant 0$

$x^T A(J,J) x = 0$ implies $X^T A X = 0$, which implies $X = 0$; and hence $x = 0$. So $A(J,J)$ is positive definite. This proves that any principal submatrix of A is positive definite. In particular all diagonal entries are positive.

Let A be positive definite. We prove that

$\det A_1 = A_1 > 0$, $\det A_2 >$, $\ldots$, $\det A_n = 0$ by induction on n. The case $n=1$ is clear. Suppose the result is true for matrices of dimension $\le n-1$. Partition A as follows

$$A = \begin{bmatrix} C & b \\ b^T & a \end{bmatrix},$$

where $A_{n-1} = C$, $a = a_{nn}$. By what we just proved $C = A_{n-1}$ is positive definite, of size $(n-1)\times(n-1)$. By induction $\det C > 0$, hence C is invertible.

Let $S = \begin{bmatrix} I & -C^{-1}b \\ 0 & 1 \end{bmatrix}$, where I is the $(n-1)\times(n-1)$ identity matrix. Then $\begin{bmatrix} C & 0 \\ 0 & a-b^TC^{-1}b \end{bmatrix} = S^TAS = \tilde{A}$.

This is the block expression of row and column operations. Notice C is symmetric, $C^T = C$, $(C^{-1})^T = C^{-1}$, and $S^T = \begin{bmatrix} I & 0 \\ -b^TC^{-1} & 1 \end{bmatrix}$.

It's easy to verify that $\tilde{A}$ is also positive definite so $a - b^TC^{-1}b > 0$ and we already know that $\det C >$. Finally

$\det \tilde{A} = \det C \cdot (a - b^T C^{-1} b) = \det A$ since $\det S = 1$. This proves $\det A > 0$ and that's all we needed to prove.

II. Suppose $\det A_1 > 0, \det A_2 > 0, \ldots, \det A_n > 0$

We want to prove that A is positive definite

Proceeding as in I, again since now by assumption, $\det C = \det A_{n-1} > 0$, we form

$$\tilde{A} = S^T A S = \begin{bmatrix} C & 0 \\ 0 & a - b^T C^{-1} b \end{bmatrix}$$

C is positive definite by inductive assumption ($C = A_{n-1}$) and $\det \tilde{A} = (\det C) \cdot (a - b^T C^{-1} b) = \det A > 0$ implies $a - b^T C^{-1} b > 0$.

Now it's easy to see that $\begin{bmatrix} C & 0 \\ 0 & a - b^T C^{-1} b \end{bmatrix}$ is positive definite

and hence so is A.

QED