Math 334 Sample Problems

One side of one notebook sized page of notes will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover up to §5.1.

1. Suppose $a < b < c < d$. Let $I = [a, b],\ J = [c, d],\ R = I \times J$ and let $f(x, y) = |x - y|$, if $x \in I,\ y \in J$. Compute $\displaystyle\int_R f$.

2. Let $P(x)$ be the parallelogram with vertices

$$(0,0), (f(x), f'(x)), (g(x), g'(x)), (f(x) + g(x), f'(x) + g'(x))$$

   where $f'' = qf, g'' = qg$ and $q(x)$ is some continuous function. Let $A(x)$ be the area of this parallelogram. Show that $A(x)$ is constant.

3. Let f be defined and bounded on $[a, b]$. Define a function g on $[a, b]$ by the formula $g(x) = \overline{I}(\chi_{[a,x]}f)$. In other words $g(x)$ is the upper integral of f on the interval $[a, x]$. Prove that g is continuous on $[a, b]$. Suppose f is continuous at x_0. Prove that $g'(x_0) = f(x_0)$. The same is true for lower integrals.

4. Let $f(x, y)$ be defined for $0 \le x \le 1, 0 \le y \le 1$ by

$$f(x,y) = \begin{cases} 1 \text{ if } x \text{ is irrational} \\ 2y \text{ if } x \text{ is rational.} \end{cases}$$

   (a) Prove that $\displaystyle\int_0^1 (\int_0^1 f(x,y)dy)dx = 1$.

   (b) What can you say about $\displaystyle\int_0^1 (\int_0^1 f(x,y)dx)dy$?

(c) Is f integrable?

5. Using the method of Lagrange multipliers, find the highest and lowest points of the circle

$$x^2 + y^2 + z^2 = 16, \ (x+1)^2 + (y+1)^2 + (z+1)^2 = 27$$

6. Let $f(x,y) = \dfrac{x}{(1+x^2+y^2)^2}$. Evaluate $\displaystyle\int_S f$, where $S = \{(x,y) : 0 \leq x \leq 2, \ 0 \leq y \leq \dfrac{x^2}{2}\}$.

7. Show that the surface $z = 3x^2 - 2xy + 2y^2$ lies entirely above every one of its tangent planes. Hint: Look at the Taylor expansion at every point.

8. Find the volume of the set

$$\{\left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1+z}\right)^2 < 1, \ -1 < z < 1\}$$

9. Let $S = \{(x,y,z) : a \leq x \leq y \leq z \leq b\}$. Prove that

$$\int_S f(x)f(y)f(z)dxdydz = \frac{1}{6}\left(\int_a^b f\right)^3$$

10. Let f be a function defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, \ \text{if } x = 0 \\ x\sin(\frac{1}{x}), \ \text{if } 0 < x \leq 1. \end{cases}$$

Prove that the curve $\{(x, f(x)) : x \in [0,1]\}$ does not have a (finite) arc length.

11. Let $f(x,y) = \sec(x+y^2)$. Find the first two non-zero terms in the Taylor series of $\cos x$, centered at 0. Use it to find the first two non-zero terms of the Taylor series of $\sec x$ centered at 0. Then use that series to find the first two non-zero terms of f at $(0,0)$.

12. Folland, problem 7, §4.2.

13. Let g be a polynomial of degree three. Prove that

$$\int_{-1}^{1} g = \frac{g(-1) + 4g(0) + g(1)}{3}.$$

    Use this formula and change of variables to find an analogous formula for $\int_a^b g$ for any a, b.

14. There may be homework problems or example problems from the text on the midterm. You may also be asked for definitions or statements of theorems, such as: implicit function theorem, Taylor's theorem, Riemann integral, fundamental theorem of calculus, area, Fubini's theorem, change of variables formula, or arc length.