

# Sample Problems

Math 334

The final exam will be held in the regular classroom from **8:30 - 10:20 a.m.** on **Monday, December 14**. You may bring one notebook size sheet of paper with notes on *both* sides. There may be homework or example problems on the final exam, in addition to problems similar to the problems on this sheet and the previous sample problem sheets. Also you should be prepared to define, state, or use the terms and theorems at the end of this sheet. The final will be comprehensive and will cover through §5.5 in Folland.

1. Suppose  $a < b < c < d$ . Let  $I = [a, b]$ ,  $J = [c, d]$ ,  $R = I \times J$  and let  $f(x, y) = |x - y|$ , if  $x \in I$ ,  $y \in J$ . Compute  $\int_R f$ .

2. Let  $P(x)$  be the parallelogram with vertices

$$(0, 0), (f(x), f'(x)), (g(x), g'(x)), (f(x) + g(x), f'(x) + g'(x))$$

where  $f'' = qf$ ,  $g'' = qg$  and  $q(x)$  is some continuous function. Let  $A(x)$  be the area of this parallelogram. Show that  $A(x)$  is constant.

3. Let  $f(x, y)$  be defined for  $0 \leq x \leq 1, 0 \leq y \leq 1$  by

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 2y & \text{if } x \text{ is rational.} \end{cases}$$

- (a) Prove that  $\int_0^1 \left( \int_0^1 f(x, y) dy \right) dx = 1$ .

- (b) What can you say about  $\int_0^1 \left( \int_0^1 f(x, y) dx \right) dy$ ?

- (c) Is  $f$  integrable?

4. Find the volume of the set

$$\left\{ \left( \frac{x}{1-z} \right)^2 + \left( \frac{y}{1+z} \right)^2 < 1, -1 < z < 1 \right\}$$

5. Let  $f$  be a  $C^1$  real-valued function on  $\mathbf{R}^1$  and define a transformation from  $\mathbf{R}^2$  to  $\mathbf{R}^2$  by the formulas  $u = f(x), v = -y + xf(x)$ . Suppose that  $f'(x_0) \neq 0$ . Show that this transformation is invertible near  $(x_0, y_0)$  for any  $y_0$ . Show that the inverse has the form  $x = g(u), y = -v + ug(u)$  for some  $C^1$  function  $g$ , defined near  $f(x_0)$ .

6. a) Show that if  $\phi$  satisfies  $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$  then

$$\nabla \cdot (\phi \nabla \phi) = |\nabla \phi|^2.$$

- b) Suppose  $\phi(x, y, z) = 3x + 2y + 4z$ . Use part a) and the divergence theorem to evaluate

$$\int \int_S \phi \frac{\partial \phi}{\partial n} dA,$$

where  $S = \{(x, y, z) : x^2 + y^2 + z^2 = 4\}$ .

7. Let  $W$  be an open set contained in  $\{(x, y, z) : x^2 + y^2 + z^2 \leq R^2\}$ . Prove that

$$\text{vol}(W) \leq \frac{R}{3} (\text{area}(\partial W))$$

8. Let  $\mathbf{F} = (y + x - x^3, 1 + x - y^3, y - 2z^3)$ . Find the surface,  $S = \partial W$ , which is the boundary of a region  $W \subset \mathbf{R}^3$ , on which  $\int_S \mathbf{F} \cdot \mathbf{n} dA$  is maximized. Compute the maximum value of this integral.

9. Find the volume of the solid bounded by the  $xy$ -plane, the cylinder  $\{(x, y, z) : x^2 + y^2 = 2x\}$ , and the cone  $\{(x, y, z) : z = \sqrt{x^2 + y^2}\}$ .

10. Let  $r = \sqrt{x^2 + y^2 + z^2}$  and let  $\mathbf{F} = (xr, yr, zr)$ . Use the divergence theorem to prove that

$$\int \int_{\partial W} \mathbf{F} \cdot \mathbf{n} dA = 4 \int \int \int_W r dV$$

By evaluating the surface integral compute

$$\int \int \int_{r \leq a} r dV$$

11. Let  $\Pi$  be the parallelotope in  $\mathbf{R}^3$  spanned by vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ . Let  $\theta_1, \theta_2, \theta_3$  be the angles between  $\mathbf{v}_1, \mathbf{v}_2; \mathbf{v}_1, \mathbf{v}_3; \mathbf{v}_2, \mathbf{v}_3$ . Prove that the volume of  $\Pi$  is the square root of

$$|\mathbf{v}_1|^2 |\mathbf{v}_2|^2 |\mathbf{v}_3|^2 (1 + 2 \cos \theta_1 \cos \theta_2 \cos \theta_3 - (\cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_3)).$$

12. Let  $S = \{(x, y, z) : a \leq x \leq y \leq z \leq b\}$ . Prove that

$$\int_S f(x)f(y)f(z) dx dy dz = \frac{1}{6} \left( \int_a^b f \right)^3$$

13. Let  $f$  be a function defined on  $[0, 1]$  by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x \sin\left(\frac{1}{x}\right), & \text{if } 0 < x \leq 1. \end{cases}$$

Prove that the curve  $\{(x, f(x)) : x \in [0, 1]\}$  does not have a (finite) arc length.

14. Let  $u$  be a function defined on  $\mathbf{R}^n$  which is homogeneous of degree  $k$ . Prove that  $\nabla^2 u$  is homogeneous of degree  $k - 2$ . Let  $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = |\mathbf{x}|$ . Compute  $\nabla^2 r^k$ .
15. Let  $S$  be the surface (torus) obtained by rotating the circle  $(x - 2)^2 + z^2 = 1$  around the  $z$ -axis. Compute the integral  $\int_S \mathbf{F} \cdot \mathbf{n} dA$ , where  $\mathbf{F} = (x + \sin(yz), y + e^{x+z}, z - x^2 \cos y)$ .
16. Compute the  $n$ -dimensional measure of the set:

$$\{(x_1, x_2, \dots, x_n) : x_j \geq 0, j = 1, \dots, n, x_1 + 2x_2 + 3x_3 + \dots + nx_n \leq n\}$$