Uniform Continuity

Note Title 10/9/2008

The statements I made in class on Thursday, Oct. 9 were wrong without severe restrictions and its best to work on a case-by-case basis. For example:

Suppose f is defined on (has domain) $[0,\infty)$. Let $X = [0,1]$ and $Y = [1,\infty)$. If $f|_X$ is continuous and $f|_Y$ is uniformly continuous, then f is uniformly continuous (on $[0,\infty)$).

Proof: X is compact, so $f|_X$ is uniformly continuous.

$\lim_{x\to 1} f_X(x) = f(1) = \lim_{x\to 1} f|_Y$ so f is continuous.

Let $\epsilon > 0$ be given. Choose $\delta > 0$ so that

(1) $|f_X(x) - f_X(y)| < \epsilon$ if $|x-y| < \delta$ and $x,y \in X$

(2) $|f_Y(x) - f_Y(y)| < \epsilon$ if $|x-y| < \delta$ and $x,y \in Y$

(3) $|f(x) - f(1)| < \epsilon/2$ if $|x-1| < \delta$

Now let $x,y \in [0,\infty)$ and suppose $|x-y| < \delta$.

If $x,y \in X$, then $|f(x) - f(y)| < \epsilon$ by (1).

If $(x,y) \in Y$, then $|f(x) - f(y)| < \epsilon$ by (2)

Suppose (say) that $x \in X$, $y \in Y$ and $|x-y| < \delta$

Then $|x-1| < \delta$ and $|y-1| < \delta$, since $x \leq 1 \leq y$, so

$|f(x) - f(1)| < \epsilon/2$ and $|f(y) - f(1)| < \epsilon/2$,

so $|f(x) - f(y)| \leq |f(x) - f(1)| + |f(y) - f(1)| < \epsilon$.

Hence f is uniformly continuous.

This result applies to $f(x) = \sqrt{x}$.