Note Title 12/2/2009

Cantor Set

The Cantor set and function illustrate many surprising properties of real numbers, functions, measures and sets. We start with a construction of the Cantor middle third set (apparently discovered five years earlier than Cantor by Smith.

Start with the unit interval $[0,1]$ and remove the open set $(1/3, 2/3)$. These numbers are all of the form $1/3 + a$, where $0 < a < 1/3$. Every number in $[0,1]$ has a ternary expansion of the form

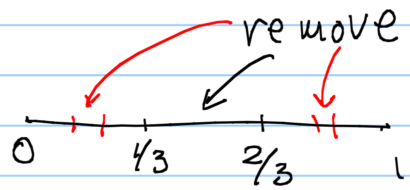
$$.t_1 t_2 t_3 \cdots \quad \text{where} \quad t_j \in \{0,1,2\}.$$

This expansion is not unique:

$$2/3 = .1222\cdots = .20\cdots0$$

The ternary expansion of any number in $(1/3, 2/3)$ <u>must</u> begin with 1 : $.1 t_2 t_3 \cdots$, $t_j \in \{0,1,2\}$.

The number $1/3 = .0222 = .1000$ has an expansion with <u>no</u> 1's.



Next remove the open middle thirds of $[0, 1/3]$ and $[2/3, 1]$.

The numbers in the middle third of $[2/3, 1]$ are of the form $2/3 + 1/9 + a$ where $0 < a < 1/9$, those in the middle third of $[0, 1/3]$ are of the form $1/9 + a$, $0 < a < 1/9$. So for example in the first case the ternary expansion must be

$$.21t_3t_4\cdots, \qquad t_j \in \{0,1,2\}$$

and hence must have a 1 in the 2nd place.

We continue with this excission and see that the numbers that remain are exactly those numbers which have no 1's in one of their ternary expansions. This is the Cantor set C.

Since it is the complement of an open set (the union of the open middle thirds) it is a closed subset of $[0,1]$ and hence compact. Its Jordan outer measure is no larger than $1-\left(1/3+2/9+4/27+\cdots \frac{1}{3}\left(\frac{2}{3}\right)^n\right)$

$$1-\frac{1}{3}\left(1+\frac{2}{3}+\cdots+\left(\frac{2}{3}\right)^n\right) = 1-\frac{1}{3}\left[\frac{1-(2/3)^{n+1}}{1/3}\right] = (2/3)^{n+1}$$

$$\longrightarrow 0$$

So $\underline{A}(C) = \overline{A}(C) = 0$. C is an uncountable set of Jordan (& Lebesgue) measure 0.

Here's the proof that C is uncountable.

Let $x = .t_1 t_2 \cdots \in C$. Then all t_j's are 0 or 2. Let $s_j = t_j/2$ and let

$y = f(x) = .s_1 s_2 \cdots$, interpreting this number as the binary expansion of a number in $[0,1]$.

Any number $y \in [0,1]$ can be obtained this way hence C is uncountable. Notice this function also is defined at any end-point of one of the removed middle thirds. For example

$$f(2/3 + 1/9) = f(.20222\ldots) = .101111 = .110\cdots0$$
$$= 3/4$$

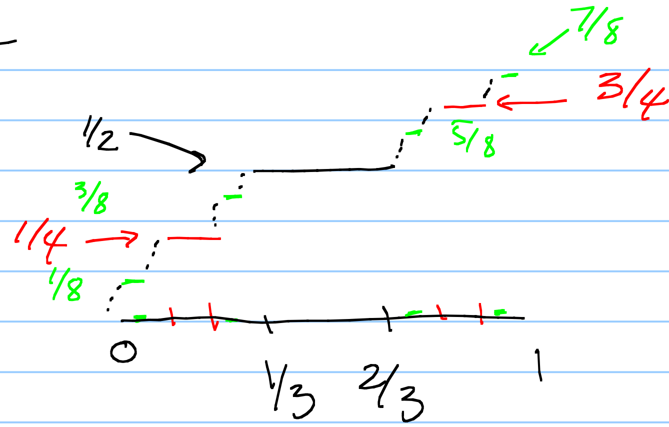
and

$$f(2/3 + 2/9) = f(.22000\ldots) = .110\cdots0 = 3/4,$$

the same value. So we extend f to be defined on all of $[0,1]$ by making it constant on the middle thirds.

Graph of

f



Now it's defined on all of $[0,1]$. f is continuous and monotone: $x > y \Rightarrow f(x) \geq f(y)$. (This must be proved.) Now we create a new function:

$h(x) = f(x) + x$. $h(x)$ is continuous and strictly increasing: $x > y \Rightarrow h(x) > h(y)$. h maps $[0,1]$ onto $[0,2]$. What is the measure of $h(C)$? It's 1.

Proof: h maps the complement of C onto a countable union of intervals of length

$$1/3 + 2/9 + 4/27 = \frac{1}{3}\left(1 + 2/3 + \cdots + (2/3)^n + \cdots\right) = \frac{1}{3}\,\frac{1}{1-2/3} = 1$$

So $\mu(h(C)) = 1$ (Lebesgue measure).

Let's summarize the properties of h:

1. Continuous

2. 1-1, increasing

3. Surjective from $[0,1]$ to $[0,2]$

4. Maps a set of measure 0 (C) onto a set $h(C)$ of measure 1.

5. $g = h^{-1}$ is continuous (Check this.)

Now let $w(x) = \begin{cases} 1 & \text{on } C \\ 0 & \text{on } [0,1] - C \end{cases}$

Then w is discontinuous exactly on C, so w is Riemann-integrable.

Let $v(y) = w(g(y))$. Then v is discontinuous exactly on $h(C)$. So v is <u>not</u> Riemann-integrable.

<u>Exercise</u>: Prove that if F is Riemann-integrable and G is continuous, then $G \circ F$ is Riemann-integrable.

<u>Remark</u>: Every set of positive measure contains a non-measurable set. Let $N \subset h(C)$ be non-measurable; then $g(N) \subset C$ has measure 0, so the image of a measurable set $(g(N))$ by a homeomorphism (h) can be non-measurable.