Lagrange Multipliers

Note Title 11/21/2009

What's the justification of the method of Lagrange multipliers? It's the implicit function theorem.

Theorem: Suppose

$$S = \{(x,y,u,v) : g(x,y,u,v)=0, h(x,y,u,v)=0\}$$

is a smooth surface, that ∇h and ∇g are linearly independent at each point of S, and that g and h are C^1.

Suppose f is differentiable in a neigborhood of S and that f restricted to S assumes a local extremum at $p=(x_0,y_0,u_0,v_0)\in S$. Then there are unique λ, μ so that

(1) $\quad \nabla f = \lambda \nabla g + \mu \nabla h \quad$ at p.

Proof: Assume $\frac{\partial(g,h)}{\partial(u,v)}(p)\neq 0$. By linear independence of ∇g and ∇h this is OK. The there are unique local solutions of $g=0, h=0$ near p of the form $u=\alpha(x,y)$, $v=\beta(x,y)$.

Let $w(x,y) = f(x,y,\alpha(x,y),\beta(x,y))$. At an extremum $\nabla w = 0$, so

(2)
$$f_x + f_u\alpha_x + f_v\beta_x = 0$$
$$f_y + f_u\alpha_y + f_v\beta_y = 0$$

We also have $g(x,y,\alpha(x,y),\beta(x,y)) = 0$, $h(x,y,\alpha(x,y),\beta(x,y)) = 0$,

(3)
$$g_x + g_u\alpha_x + g_v\beta_x = 0, \quad g_y + g_u\alpha_y + g_v\beta_y = 0$$
$$h_x + h_u\alpha_x + h_v\beta_x = 0, \quad h_y + h_u\alpha_y + h_v\beta_y = 0$$

Now the vectors (g_u, g_v), (h_u, h_v) are linearly independent.

So there are unique λ and μ so that

(4)
$$f_u = \lambda g_u + \mu h_u, \quad f_v = \lambda g_v + \mu h_v.$$

By (2) $f_x = -(f_u\alpha_x + f_v\beta_x)$,

by (4) $= -[(\lambda g_u + \mu h_u)\alpha_x + (\lambda g_v + \mu h_v)\beta_x]$

$= -\lambda[g_u\alpha_x + g_v\beta_x] - \mu[h_u\alpha_x + h_v\beta_x]$

by (3) $= -\lambda(-g_x) - \mu(-h_x)$

$$f_x = \lambda g_x + \mu h_x$$

Similarly $f_y = \lambda g_y + \mu h_y$.

Therefore $\nabla f = \lambda \nabla g + \mu \nabla h$.