Jensen's Integral Inequality

Note Title 11/27/2009

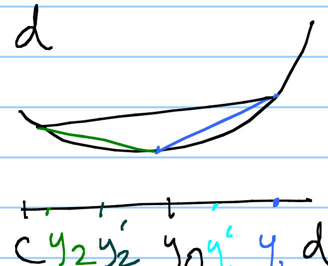
I'm going to state and prove a simple version of Jensen's integral inequality for convex functions. There is a more general result. Its proof is an adaptation of this proof.

<u>Theorem</u>: Let φ be a differentiable convex function defined on $[c, d]$. Let $p(x) \geq 0$ be a continuous function on $[a, b]$ such that $\int_a^b p = 1$ (a probability density). Let f be a continuous function on $[a, b]$ and let such that $c \leq f(x) \leq d$, and let $\bar{f} = \int_a^b p(x) f(x)\,dx$ be its average. Then

$$(1) \qquad \varphi(\bar{f}) \leq \int_a^b p(x)\,\varphi(f(x))\,dx$$

" φ (average f) $\leq$ average $(\varphi \circ f)$ ".

<u>Proof</u> Let $y_0 = \bar{f}$, $c \leq y_0 \leq d$



$$y_0 = \frac{1}{(y_1 - y_2)}\left[(y_1 - y_0) y_2 + (y_0 - y_2) y_1\right]$$

$$= \left(\frac{y_1 - y_0}{y_1 - y_2}\right) y_2 + \frac{(y_0 - y_2)}{(y_1 - y_2)} y_1 .$$

So $y_0 = p_2 y_2 + p_1 y_1$, where $p_2 = \frac{y_1 - y_0}{y_1 - y_2}$, $p_1 = \frac{y_0 - y_2}{y_1 - y_2}$

$p_1 \geq 0$, $p_2 \geq 0$, $p_1 + p_2 = 1$.

Hence

$$\varphi(y_0) \leq \frac{1}{(y_1 - y_2)} \left[(y_1 - y_0)\varphi(y_2) + (y_0 - y_2)\varphi(y_1) \right],$$

$$\varphi(y_0)(y_1 - y_0 + y_0 - y_2) \leq (y_1 - y_0)\varphi(y_2) + (y_0 - y_2)\varphi(y_1)$$

$$(\varphi(y_0) - \varphi(y_2))(y_1 - y_0) \leq (\varphi(y_1) - \varphi(y_0))(y_0 - y_2)$$

Or $$\frac{\varphi(y_0) - \varphi(y_2)}{(y_0 - y_2)} \leq \frac{\varphi(y_1) - \varphi(y_0)}{(y_1 - y_0)}$$

A similar argument proves that if $y_2 < y_2' < y_0 < y_1' < y_1$

$$\frac{\varphi(y_0) - \varphi(y_2)}{y_0 - y_2} \leq \frac{\varphi(y_0) - \varphi(y_2')}{y_0 - y_2'} \leq \frac{\varphi(y_1') - \varphi(y_0)}{y_1' - y_0} \leq \frac{\varphi(y_1) - \varphi(y_0)}{y_1 - y_0}$$

So the right difference quotients derease to $\varphi'(y_0)$ and the left difference quotients increase to $\varphi'(y_0)$.

i.e. $$\frac{\varphi(y_0) - \varphi(y_2)}{y_0 - y_2} \leq \varphi'(y_0) \leq \frac{\varphi(y_1) - \varphi(y_0)}{y_1 - y_0}.$$

Finally $\varphi(y_1) \geq \varphi'(y_0)(y_1 - y_0) + \varphi(y_0)$, if $y_1 > y_0$

and $\varphi(y_2) \geq \varphi'(y_0)(y_2 - y_0) + \varphi(y_0)$, if $y_2 < y_0$.

This says $z = \varphi(y)$ is always above the tangent line

$$z = \varphi'(y_0)(y - y_0) + \varphi(y_0)$$

We have

$$\varphi(y) \geq \varphi'(y_0)(y-y_0) + \varphi(y_0)$$

Let $y = f(x)$:

$$\varphi(f(x)) \geq \varphi'(y_0)(f(x)-y_0) + \varphi(y_0),$$

and multiply by $p(x) \geq 0$ and then integrate:

$$\int_a^b p(x)\,\varphi(f(x))\,dx \geq \varphi'(y_0)\int_a^b p(x) f(x)\,dx - \varphi'(y_0)\cdot y_0 \int_a^b p(x)\,dx$$
$$+ \varphi(y_0)\int_a^b p(x)\,dx.$$

Use $\int p = 1$ to get, and $y_0 = \int_a^b p(x) f(x)\,dx$

$$\int_a^b p(x)\,\varphi(f(x))\,dx \geq \varphi'(y_0)\cdot y_0 - \varphi'(y_0)\cdot y_0 + \varphi(y_0),$$

to get (1).