Math 334 Sample Problems

One notebook-sized page (one side) of notes will be allowed on the test. You may work together on the sample problems – I encourage you to do that. Neither Will nor I will tell you how to work these problems. You may ask for help if you bring your work in and show it to us, and tell us where you are stuck. The test (on Wednesday, October 27) will cover up to and including section 2.5 in the text. There may be homework problems or example problems from the text on the midterm. There will be four problems on the midterm, at least one will be from this list or a slight modification of a problem on this list. This list is long to give a broad sample of the types of problems.

1. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} (1 - \cos\frac{x^2}{y})\sqrt{x^2 + y^2} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

   (a) Prove that f is continuous at $(0,0)$.

   (b) Compute all directional derivatives at $(0,0)$.

   (c) Prove that f is not differentiable at $(0,0)$.

2. (a) Let f be a continuous function from $[0,1]$ to $\mathbf{R}^n$. Prove that the graph of f, $\Gamma_f = \{(x,y) : y = f(x)\}$ is closed. (Γ_f is a subset of $[0,1] \times \mathbf{R}^n$.)

   (b) Let f be a function from $[0,1]$ to $\mathbf{R}^n$. Suppose Γ_f is closed. Is f continuous? Give a proof or counterexample.

3. Let $A \subset \mathbf{R}^n$ and $B \subset \mathbf{R}^m$ be compact. Prove that $A \times B \subset \mathbf{R}^{n+m}$ is compact.

4. Let $\mathbb{Q}$ be the set of rational numbers. Prove that there does NOT exist a pair of non-empty sets A, B such that $\mathbb{Q} = A \cup B$ with $\overline{A} \cap \overline{B} = \emptyset$.

5. Prove that the temperature of a tetrahedron must have at least three distinct points on the edges or vertices of the tetrahedron with the same value. Assume the temperature is a continuous function. (Hint: the intermediate value theorem.)

6. Let f be defined as follows

$$f(x, y) = \begin{cases} x^{4/3} \sin(y/x) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0. \end{cases}$$

Where is f differentiable?

7. Let a sequence be defined recursively by the rules: $x_0 = 1, x_{n+1} = x_n + \dfrac{1}{x_n}$. Prove that the sequence does not converge.

8. Suppose $\{a_n\}$ is a sequence with $a_n > 0$ and b_n is defined by $b_n = a_n + \dfrac{1}{a_n}$.

   (a) Assume that $a_n \geq 1$ and that $\{b_n\}$ converges. Prove that $\{a_n\}$ converges.

   (b) If it is assumed that $\{b_n\}$ converges but only that $a_n > 0$, it does not necessarily follow that $\{a_n\}$ converges. Find such an example.

9. Prove that $\lim\limits_{n \to \infty} \sin n$ does not exist.

10. Prove that the set $\{(x, y, z) : \dfrac{x^2}{2} + \dfrac{y^2}{3} + \dfrac{z^2}{4} = 1\}$ is connected.

11. Suppose A is a connected set and that $A \subset B \subset \overline{A}$. Prove that B is connected.

12. Let f be continuous on $(0,1)$ and suppose $0 < f(x) < x$ for all $x \in (0,1)$. Define $f_n(x)$ inductively by $f_1(x) = f(x), f_{n+1}(x) = f(f_n(x))$. Prove that $\lim_{n\to\infty} f_n(x)$ exists and compute it.

13. Let $|a| < 1$, where a is a real number. Prove that $\lim_{n\to\infty} na^n = 0$. Notice that a is allowed to be negative.

14. Let f be a continuous real valued function defined on $[0,1]$ such that $f(0) = f(1)$. Show that there is a pair of points $a, b \in [0,1]$ such that $b - a = 1/2$ and $f(b) = f(a)$.

15. Let $\{x_n\}_1^\infty$ be a sequence of real numbers. Prove that there exists a sequence of intervals $I_n = [a_n, b_n], I_{n+1} \subset I_n$ with $x_n \notin I_n$. Use this to prove that there is a real number that is not in the sequence $\{x_n\}_1^\infty$. Thus prove that the reals are not countable.

16. You will need to know the the following items

    (a) Cauchy's inequality
    (b) Triangle inequality
    (c) Open set
    (d) Closed set
    (e) Boundary of a set
    (f) Compact set
    (g) Bolzano-Weierstrass theorem
    (h) Connected set
    (i) Convergent sequence
    (j) Completeness
    (k) Cauchy's criterion
    (l) Continuity at a point
    (m) Continuity on a set
    (n) Uniform continuity
    (o) Partial derivatives
    (p) Differentiability
    (q) Mean value theorem
    (r) Chain rule