Sample Problems

Math 334

The final exam will be held in the regular classroom from **8:30 - 10:20 a.m.** on **Monday, December 13** in Sieg 226. You may bring one notebook size sheet of paper with notes on *both* sides. There may be homework or example problems on the final exam, in addition to problems similar to the problems on this sheet and the previous sample problem sheets. Also you should be prepared to define, state, or use the terms and theorems at the end of this sheet. The final will be comprehensive and will cover through §5.3 in Folland.

1. Let $P(x)$ be the parallelogram with vertices

$$(0,0),(f(x),f'(x)),(g(x),g'(x)),(f(x)+g(x),f'(x)+g'(x))$$

where $f''=qf, g''=qg$ and $q(x)$ is some continuous function. Let $A(x)$ be the area of this parallelogram. Show that $A(x)$ is constant.

2. Let $f(x,y)$ be defined for $0\le x\le 1, 0\le y\le 1$ by

$$f(x,y)=\begin{cases}1 \text{ if } x \text{ is irrational}\\ 2y \text{ if } x \text{ is rational.}\end{cases}$$

   (a) Prove that $\int_0^1 (\int_0^1 f(x,y)dy)dx = 1$.

   (b) What can you say about $\int_0^1 (\int_0^1 f(x,y)dx)dy$?

   (c) Is f integrable?

3. Find the volume of the set

$$\{\left(\frac{x}{1-z}\right)^2+\left(\frac{y}{1+z}\right)^2<1,\ -1<z<1\}$$

4. Let f be a C^1 real-valued function on $\mathbf{R}^1$ and define a transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ by the formulas $u=f(x), v=-y+xf(x)$. Suppose that $f'(x_0)\ne 0$. Show that this transformation is invertible near (x_0,y_0) for any y_0. Show that the inverse has the form $x=g(u), y=-v+ug(u)$ for some C^1 function g, defined near $f(x_0)$.

5. Find the volume of the solid bounded by the xy-plane, the cylinder $\{(x,y,z): x^2+y^2=2x\}$, and the cone $\{(x,y,z): z=\sqrt{x^2+y^2}\}$.

6. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ x^2 + x \text{ if } x \notin \mathbb{Q} \end{cases}$$

Prove that $f'(0)$ exists and $f'(0) > 0$, but that f is not invertible near 0. Why does this not contradict the inverse function theorem?

7. Let Π be the parallelotope in $\mathbf{R}^3$ spanned by vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. Let $\theta_1, \theta_2, \theta_3$ be the angles between $\mathbf{v_1}, \mathbf{v_2}; \mathbf{v_1}, \mathbf{v_3}; \mathbf{v_2}, \mathbf{v_3}$. Prove that the volume of Π is the square root of

$$|\mathbf{v_1}|^2|\mathbf{v_2}|^2|\mathbf{v_3}|^2 \left(1 + 2\cos\theta_1\cos\theta_2\cos\theta_3 - (\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3)\right).$$

8. Let $S = \{(x, y, z) : a \le x \le y \le z \le b\}$. Prove that

$$\int_S f(x)f(y)f(z)dxdydz = \frac{1}{6}\left(\int_a^b f\right)^3$$

9. Let f be a function defined on $[0, 1]$ by

$$f(x) = \begin{cases} 0, \text{ if } x = 0 \\ x\sin(\frac{1}{x}), \text{ if } 0 < x \le 1. \end{cases}$$

Prove that the curve $\{(x, f(x)) : x \in [0, 1]\}$ is not rectifiable.

10. Let u be a function defined on $\mathbf{R}^n$ which is homogeneous of degree k. Prove that $\nabla^2 u$ is homogeneous of degree $k-2$. Let $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = |\mathbf{x}|$. Compute $\nabla^2 r^k$.

11. Let S be the surface (torus) obtained by rotating the circle $(x-2)^2 + z^2 = 1$ around the z-axis. Compute the integral $\int_S \mathbf{F} \cdot \mathbf{n} dA$, where $\mathbf{F} = (x + \sin(yz), y + e^{x+z}, z - x^2\cos y)$.

12. Compute the n-dimensional measure of the set:

$$\{(x_1, x_2, \ldots, x_n) : x_j \ge 0, j = 1, \ldots, n,\ x_1 + 2x_2 + 3x_3 + \cdots + nx_n \le n\}$$

13. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuously differentiable. Suppose $|f_x(x, y)| \le K,\ |f_y(x, y)| \le K$ for all (x, y). Prove that

$$|f(x_1, y_1) - f(x_2, y_2)| \le \sqrt{2}K\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

14. Additional definitions, terms, and theorems: Jacobians, arc length formula, unit tangent to a parameterized curve, unit normal at the boundary of a region in two or three space, line integral, surface area formula, Green's theorem, surface integrals, Green's formula.