Sample Problems

Math 334

The final exam will be held in the regular classroom from **8:30 - 10:20 a.m.** on **Monday, December 12** in MEB 238. You may bring one notebook size sheet of paper with notes on *both* sides. There may be homework or example problems on the final exam, in addition to problems similar to the problems on this sheet and the previous sample problem sheets. Also you should be prepared to define, state, or use the terms and theorems at the end of this sheet. The final will be comprehensive and will cover through §5.5 in Folland.

1. Suppose $a < b < c < d$. Let $I = [a, b]$, $J = [c, d]$, $R = I \times J$ and let $f(x, y) = |x - y|$, if $x \in I$, $y \in J$. Compute $\int_R f$.

2. Let $P(x)$ be the parallelogram with vertices

$$(0,0), (f(x), f'(x)), (g(x), g'(x)), (f(x) + g(x), f'(x) + g'(x))$$

   where $f'' = qf, g'' = qg$ and $q(x)$ is some continuous function. Let $A(x)$ be the area of this parallelogram. Show that $A(x)$ is constant.

3. Let I be an interval in $\mathbb{R}$. I might be open, closed, or neither. Let f be a real valued continuous function defined on I. Suppose f as no local maxima or minima in the interior of $\mathbb{R}$. Then prove that f is monotonic.

4. Let I be an interval in $\mathbb{R}$. I might be open, closed, or neither. Suppose $f : I \to \mathbb{R}$ is strictly increasing. Prove that if the image of f is connected then f is continuous.

5. Let $f(x, y)$ be defined for $0 \leq x \leq 1, 0 \leq y \leq 1$ by

$$f(x, y) = \begin{cases} 1 \text{ if } x \text{ is irrational} \\ 2y \text{ if } x \text{ is rational.} \end{cases}$$

   (a) Prove that $\int_0^1 (\int_0^1 f(x, y) dy) dx = 1$.

   (b) What can you say about $\int_0^1 (\int_0^1 f(x, y) dx) dy$?

   (c) Is f integrable?

6. Find the volume of the set

$$\left\{ \left(\frac{x}{1-z} \right)^2 + \left(\frac{y}{1+z} \right)^2 < 1, \ -1 < z < 1 \right\}$$

7. Let f be a C^1 real-valued function on $\mathbf{R}^1$ and define a transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ by the formulas $u = f(x), v = -y + xf(x)$. Suppose that $f'(x_0) \neq 0$. Show that this transformation is invertible near (x_0, y_0) for any y_0. Show that the inverse has the form $x = g(u), y = -v + ug(u)$ for some C^1 function g, defined near $f(x_0)$.

8. Find the volume of the solid bounded by the xy-plane, the cylinder $\{(x, y, z) : x^2 + y^2 = 2x\}$, and the cone $\{(x, y, z) : z = \sqrt{x^2 + y^2}\}$.

9. Let Π be the parallelotope in $\mathbf{R}^3$ spanned by vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. Let $\theta_1, \theta_2, \theta_3$ be the angles between $\mathbf{v_1}, \mathbf{v_2}; \mathbf{v_1}, \mathbf{v_3}; \mathbf{v_2}, \mathbf{v_3}$. Prove that the volume of Π is the square root of

$$|\mathbf{v_1}|^2 |\mathbf{v_2}|^2 |\mathbf{v_3}|^2 \left(1 + 2 \cos \theta_1 \cos \theta_2 \cos \theta_3 - (\cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_3) \right).$$

10. Let $S = \{(x, y, z) : a \leq x \leq y \leq z \leq b\}$. Prove that

$$\int_S f(x)f(y)f(z)dxdydz = \frac{1}{6} \left(\int_a^b f \right)^3$$

11. Let f be a function defined on $[0, 1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x \sin(\frac{1}{x}), & \text{if } 0 < x \leq 1. \end{cases}$$

Prove that the curve $\{(x, f(x)) : x \in [0, 1]\}$ is not rectifiable.

12. Let u be a function defined on $\mathbf{R}^n$ which is homogeneous of degree k. Prove that $\nabla^2 u$ is homogeneous of degree $k - 2$. Let $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = |\mathbf{x}|$. Compute $\nabla^2 r^k$.

13. Let S be the surface (torus) obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ around the z-axis. Compute the integral $\int_S \mathbf{F} \cdot \mathbf{n} dA$, where $\mathbf{F} = (x + \sin(yz), y + e^{x+z}, z - x^2 \cos y)$.

14. Compute the n-dimensional measure of the set:

$$\{(x_1, x_2, \ldots, x_n) : x_j \geq 0, j = 1, \ldots, n, \ x_1 + 2x_2 + 3x_3 + \cdots + nx_n \leq n\}$$

15. Additional definitions, terms, and theorems: Jacobians, Divergence, Arc length formula, Unit tangent to a parameterized curve, Surface area formula, Green's theorem, Surface integrals, Divergence theorem, Green's identities.