Math 334 Sample Problems

One side of one notebook sized page of notes will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover up to §4.4.

1. Expand $(1-x+2y)^3$ in powers of $x-1$ and $y-2$ in two different ways. The first way is by using algebra and the second way is by computing the Taylor series.

2. Let f be defined and bounded on $[a,b]$. Define a function g on $[a,b]$ by the formula $g(x) = \overline{I}(\chi_{[a,x]}f)$. In other words $g(x)$ is the upper integral of f on the interval $[a,x]$. Prove that g is continuous on $[a,b]$. Suppose f is continuous at x_0. Prove that $g'(x_0) = f(x_0)$. The same is true for lower integrals.

3. Folland, §4.2, # 7.

4. Suppose $a < b < c < d$. Let $I = [a,b],\ J = [c,d],\ R = I \times J$ and let $f(x,y) = |x-y|$, if $x \in I,\ y \in J$. Compute $\int_R f$.

5. Using the method of Lagrange multipliers, find the highest and lowest points of the circle

$$x^2 + y^2 + z^2 = 16,\ (x+1)^2 + (y+1)^2 + (z+1)^2 = 27$$

6. Show that the surface $z = 3x^2 - 2xy + 2y^2$ lies entirely above every one of its tangent planes. Hint: Look at the Taylor expansion at every point.

7. Compute the volume of the region bounded by the two cylinders $x^2 + y^2 = a^2,\ x^2 + z^2 = a^2$.

8. Find the shortest distance from the point $(1, -1, 1)$ to the surface $z = xy$.

9. Let f be continuously differentiable on $[a, b]$ and assume $f'(x) > 0$ on $[a, b]$.

   (a) Prove that f has a continously differentiable inverse g and that $g'(x) > 0$.

   (b) Prove that
   $$\int_a^b f + \int_{f(a)}^{f(b)} g = bf(b) - af(a).$$
   Can you give a geometric interpretation of the result?

10. Let $f(x, y) = \sec(x + y^2)$. Find the first two non-zero terms in the Taylor series of $\cos x$, centered at 0. Use it to find the first two non-zero terms of the Taylor series of $\sec x$ centered at 0. Then use that series to find the first two non-zero terms of f at $(0, 0)$.

11. Let g be a polynomial of degree three. Prove that
$$\int_{-1}^{1} g = \frac{g(-1) + 4g(0) + g(1)}{3}.$$

12. Consider the following function
$$F(x, y) = \left(\frac{x}{1 + x + y}, \frac{y}{1 + x + y}\right),$$
which has the set $\{(x, y) : 1 + x + y \neq 0\}$ as its domain. Compute $\dfrac{\partial(f, g)}{\partial(x, y)}$. Where is it different from 0? Show that F is $1 - 1$ and find an explicit formula for its inverse. Use these results to describe the exact image of F

13. Folland, §2.9, problem 16.

14. Let f be a positive continuous function on $I = [a, b]$. Let $M = \max\{f(x) : x \in I\}$. Prove that

$$\lim_{n\to\infty} \left(\int_I f^n \right)^{1/n} = M.$$

15. Suppose $F(x, y)$ is a C^2 function that satisfies the equations $F(x, y) = F(y, x), F(x, x) = x$. Prove that the quadratic term in the Taylor polynomial of F based at the point (a, a) is $\frac{1}{2} F_{xx}(a, a)(x - y)^2$.

16. There may be homework problems or example problems from the text or lectures on the midterm.