Jensen Inequality

Theorem 1. *Let f be an integrable function defined on $[a, b]$ and let ϕ be a continuous (this is not needed) convex function defined at least on the set $[m, M]$ where m is the int of f and M is the sup of f. Then*

$$\phi(\frac{1}{b-a}\int_a^b f) \leq \frac{1}{b-a}\int_a^b \phi(f).$$

Proof. We take the following definition of a convex function. ϕ is convex if for every point $(x_0, \phi(x_0))$ on the graph of ϕ there is a line $y = \alpha(x - x_0) + \phi(x_0)$ such that $\phi(x) \geq \alpha(x - x_0) + \phi(x_0)$ for all x in the domain of ϕ. Now let $x_0 = \frac{1}{b-a}\int_a^b f$ and integrate the inequality

$$\phi(f(x)) \geq \alpha(f(x) - x_0) + \phi(x_0).$$

We get

$$\int \phi(f) \geq \alpha(x_0 - x_0)(b-a) + (b-a)\phi(x_0) = (b-a)\phi(\frac{1}{b-a}\int_a^b f),$$

which is what we want. This is much easier to remember if $b - a = 1$:

$$\phi(\int f) \leq \int \phi(f).$$

Restated:

$$\phi(average(f)) \leq average\ \phi(f).$$

□