Math 334 Sample Problems

One side of one notebook sized page of notes will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover up to §3.4.

1. Suppose that (x_0, y_0, z_0, u_0) satisfies the equations

$$\begin{aligned} x + y + z &= F(u) \\ x^2 + y^2 + z^2 &= G(u) \\ x^3 + y^3 + z^3 &= H(u), \end{aligned}$$

   where, F, G, H are C^1 in a neighborhood of u_0. State a sufficient condition for being able to solve these equations for x, y, z as C^1 functions of u in a neighborhood of (x_0, y_0, z_0, u_0).

2. Is the set $\{(x, y) : y^2 + x^2 e^y = 0\}$ a smooth curve? Is the set $\{(a \cos t, b \sin t) : t \in (0, \pi)\}$, where $a > 0, b > 0$ a smooth curve?

3. Expand $(1 - x + 2y)^3$ in powers of $x - 1$ and $y - 2$ in two different ways. The first way is by using algebra and the second way is by computing the Taylor series.

4. Using the method of Lagrange multipliers, find the highest and lowest points of the circle

$$x^2 + y^2 + z^2 = 16, \ (x + 1)^2 + (y + 1)^2 + (z + 1)^2 = 27$$

5. Show that the surface $z = 3x^2 - 2xy + 2y^2$ lies entirely above every one of its tangent planes. Hint: Look at the Taylor expansion at every point.

6. Let $a > 0$ and $b > 0$. Decide whether or not the map $F(r, t) = (ra \cos t, rb \sin t)$ from $\{(r, t) : 0 < r < 1, 0 < t < \pi/2\}$ to $\{(x, y) : x > 0, y > 0, \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} < 1\}$ has a differentiable inverse.

7. Let $a > 0, b > 0$ and $a + b = 1$. Also let $x > 0, y > 0$. prove that

$$x^a y^b \leq ax + by,$$

   by using the method of Lagrange multipliers applied to maximize $x^a y^b$ subject to $ax + by = c$, where $c > 0$ is some constant.

8. Let $f(x,y) = \sec(x+y^2)$. Find the first two non-zero terms in the Taylor series of $\cos x$, centered at 0. Use it to find the first two non-zero terms of the Taylor series of $\sec x$ centered at 0. Then use that series to find the first two non-zero terms of f at $(0,0)$.

9. Define $f(x) = (\log x)^{\log x}$, for $x > 1$. Using the chain rule, compute $f'(x)$.

10. Consider the following function

$$F(x,y) = (\frac{x}{1+x+y}, \frac{y}{1+x+y}),$$

which has the set $\{(x,y) : 1+x+y \neq 0\}$ as its domain. Compute $\dfrac{\partial(f,g)}{\partial(x,y)}$. Where is it different from 0? Show that F is $1-1$ and find an explicit formula for its inverse. Use these results to describe the exact image of F

11. Folland, §2.9, problem 16(a).

12. Suppose $F(x,y)$ is a C^2 function that satisfies the equations $F(x,y) = F(y,x), F(x,x) = x$. Prove that the quadratic term in the Taylor polynomial of F based at the point (a,a) is $\frac{1}{2}F_{xx}(a,a)(x-y)^2$.

13. There may be homework problems or example problems from the text or lectures on the midterm.

14. The following topics have been covered since the first midterm:
    (a) Higher order partials and equality of mixed partials.
    (b) Taylor's theorem in one and several variables with Lagrange's form of the remainder.
    (c) Behavior near critical points – second derivative test for extrema in the case of two variables.
    (d) Max-min problems with constraints. The method of Lagrange multipliers.
    (e) Implicit and Inverse Function Theorems.
    (f) Smooth curves and surfaces.
    (g) Jacobians and their uses in change of variables and computing derivatives.