A Polynomial With n Maxima and no Other Critical Points

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Theorem 1 *Let*

$$h = \prod_{i=1}^{n-1} (x - 2i),$$

and let

$$g = \int \prod_{i=1}^{2n-1} (x - i) dx.$$

Then $f = (yh^2 - 2x - 1)^2 + g$ *has exactly* n *minima, and no other critical affine real critical points.*

Proof: We compute the partials of f to see that

$$\begin{aligned} f_x &= 2(2yh_x h - 2)(yh^2 - 2x - 1) + g_x \\ f_y &= 2h^2(yh^2 - 2x - 1). \end{aligned}$$

If $f_y = 0$, then either $h = 0$, or $yh^2 - 2x - 1 = 0$. In the former case, we see that $f_x = 4(2x + 1) \neq 0$. Thus, if (x, y) is a critical point, then $yh^2 - 2x - 1 = 0$. Assuming this, we have $f_x = g_x$, so $x \in \{1, \ldots, 2n - 1\}$. But $y = (2x + 1)/h^2$ is undefined if $x \in \{2, 4, \ldots, 2n - 2\}$, so the only critical points of f are $(x, (2x + 1)/h^2)$, where $x \in \{1, 3, \ldots, 2n - 1\}$.

We can now determine the types of these critical points. Assuming that $yh^2 - 2x - 1 = 0$, computation reveals

$$\begin{aligned} f_{xx} &= 2(2yh_xh - 2)^2 + g_{xx} \\ f_{xy} = f_{yx} &= 2h^2(2yh_xh - 2) \\ f_{yy} &= 2h^4, \end{aligned}$$

so

$$Hf = f_{xx}f_{yy} - f_{xy}^2 = 2h^4 g_{xx}.$$

Since $h \neq 0$ for $x \in \{1, 3, \ldots, 2n-1\}$, it suffices to show that $g_{xx} > 0$ in order to show that all of the critical points are extrema. But this can be seen by observing that g_x is of degree $2n-1$ and has $2n-1$ zeros. Since the coefficient of x^{2n-1} in g_x is 1, it follows that g_{xx} must have sign $(+, -, +, \ldots, -, +)$ for the zeros of g_x. In particular, for $x \in \{1, 3, \ldots, 2n-1\}$, we have $g_{xx} > 0$.

Finally, observe that $f_{yy} \geq 0$, so that the extrema are in fact minima. □