Tonelli's Theorem

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There is no reasonable statement of Tonelli's theorem for Riemann integrals. The Lebesgue integral statement is

Theorem 1. *Suppose $f(x, y)$ is Lebesgue measurable on a rectangle $R = I_1 \times I_2$ and $f \geq 0$. Then*

$$\int_R f = \int_{I_1} \left(\int_{I_2} f(x, y)dy \right) dx$$

What does this mean? It means that $f(x, y)$ is Lebesgue measurable as a function of y for each fixed x, that $\displaystyle\int_{I_2} f(x, y)dy$ is a measurable *extended* real valued function (possibly assuming the value $+\infty$) which is Lebesgue measurable and that its integral (possibly $+\infty$) is equal to $\int_R f$.

There is no reasonable Riemann integral statement of this theorem (no reasonable definition of a Jordan measurable function).

An attempt might be

Theorem 2 (False)**.** *Suppose $f \geq 0$ and suppose*

$$\int_{I_1} \left(\int_{I_2} f(x, y)dy \right) dx = \int_{I_2} \left(\int_{I_1} f(x, y)dx \right) dy.$$

Then

$$\int_R f = \int_{I_1} \left(\int_{I_2} f(x, y)dy \right) dx.$$

Here is a counter example, taken from Apostol's book. Define a set $Q = \cup_{j=1}^{\infty} R_j$, where $R_j = \{(\frac{m}{p_j}, \frac{n}{p_j}) : 0 < m, n < p_j - 1\}$ and p_j is the j^{th} prime. Let

$$f(x, y) = \begin{cases} 0, & \text{if } (x, y) \in Q \\ 1 & \text{if } (x, y) \notin Q \end{cases}$$

Then

$$\int_0^1 \left(\int_0^1 f(x, y)dy \right) dx = 1 = \int_0^1 \left(\int_0^1 f(x, y)dx \right) dy$$

But f is not Riemann integrable; f is Lebesgue integrable and its integal is 1 as it should be.