EXERCISES

1. For each of the following sets S in the plane $\mathbb{R}^2$, do the following: (i) Draw a sketch of S. (ii) Tell whether S is open, closed, or neither. (iii) Describe S^{int}, $\overline{S}$, and ∂S. (These descriptions should be in the same set-theoretic language as the description of S itself given here.)
   a. $S = \{(x, y) : 0 < x^2 + y^2 \leq 4\}$.
   b. $S = \{(x, y) : x^2 - x \leq y \leq 0\}$.
   c. $S = \{(x, y) : x > 0,\ y > 0, \text{ and } x + y > 1\}$.
   d. $S = \{(x, y) : y = x^3\}$.
   e. $S = \{(x, y) : x > 0 \text{ and } y = \sin(1/x)\}$.
   f. $S = \{(x, y) : x^2 + y^2 < 1\} \setminus \{(x, 0) : x < 0\}$.
   g. $S = \{(x, y) : x \text{ and } y \text{ are rational numbers in } [0, 1]\}$.
2. Show that for any $S \subset \mathbb{R}^n$, S^{int} is open and ∂S and $\overline{S}$ are both closed. (*Hint:* Use the fact that balls are open, proved in Example 1.)
3. Show that if S_1 and S_2 are open, so are $S_1 \cup S_2$ and $S_1 \cap S_2$.
4. Show that if S_1 and S_2 are closed, so are $S_1 \cup S_2$ and $S_1 \cap S_2$. (One way is to use Exercise 3 and Proposition 1.4b.)
5. Show that the boundary of S is the intersection of the closures of S and S^c.
6. Give an example of an infinite collection $S_1, S_2, \ldots$ of closed sets whose union $\bigcup_{j=1}^{\infty} S_j$ is not closed.
7. There are precisely two subsets of $\mathbb{R}^n$ that are both open and closed. What are they?
8. Give an example of a set S such that the interior of S is unequal to the interior of the closure of S.
9. Show that the ball of radius r about $\mathbf{a}$ is contained in the ball of radius $r + |\mathbf{a}|$ about the origin. Conclude that a set $S \subset \mathbb{R}^n$ is bounded if it is contained in some ball (whose center can be anywhere in $\mathbb{R}^n$).

1.3 Limits and Continuity

We now commence our study of functions defined on $\mathbb{R}^n$ or subsets of $\mathbb{R}^n$. For the most part we shall be dealing with real-valued functions, but in many situations we shall deal with vector-valued or complex-valued functions, that is, functions whose values lie in $\mathbb{R}^k$ or $\mathbb{C}$. For our present purposes we can regard $\mathbb{C}$ as $\mathbb{R}^2$ by identifying the complex number $u + iv$ with the ordered pair (u, v), so it is enough to consider vector-valued functions. But we begin with the real-valued case.

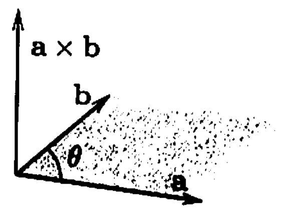

FIGURE 1.1: The geometry of the cross product.

in other words, $\mathbf{a} \times \mathbf{b}$ *is orthogonal to both* $\mathbf{a}$ *and* $\mathbf{b}$. See Figure 1.1.

The two italicized statements specify the magnitude and direction of $\mathbf{a} \times \mathbf{b}$ in purely geometric terms and show that $\mathbf{a} \times \mathbf{b}$ has an intrinsic geometric meaning, independent o. the choice of coordinate axes. Well, almost: The fact that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ specifies its direction only up to a factor of ± 1, and this last bit of information is provided by the "right hand rule": If you point the thumb and first finger of your right hand in the directions of $\mathbf{a}$ and $\mathbf{b}$, respectively, and bend the middle finger so that it is perpendicular to both of them, the middle finger points in the direction of $\mathbf{a} \times \mathbf{b}$. Thus the definition of cross product is tied to the convention of using "right-handed" coordinate systems. If we were to switch to "left-handed" ones, all cross products would be multiplied by -1.

EXERCISES

1. Let $\mathbf{x} = (3, -1, -1, 1)$ and $\mathbf{y} = (-2, 2, 1, 0)$. Compute the norms of $\mathbf{x}$ and $\mathbf{y}$ and the angle between them.
2. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, show that
   a. $|\mathbf{x} + \mathbf{y}|^2 = |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2$.
   b. $|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2(|\mathbf{x}|^2 + |\mathbf{y}|^2)$.
3. Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$.
   a. Generalize Exercise 2a to obtain a formula for $|\mathbf{x}_1 + \cdots + \mathbf{x}_k|^2$.
   b. (*The Pythagorean Theorem*) Suppose the vectors $\mathbf{x}_j$ are mutually orthogonal, i.e., that $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ for $i \neq j$. Show that $|\mathbf{x}_1 + \cdots + \mathbf{x}_k|^2 = |\mathbf{x}_1|^2 + \cdots + |\mathbf{x}_k|^2$.
4. Under what conditions on $\mathbf{a}$ and $\mathbf{b}$ is Cauchy's inequality an equality? (Examine the proof.)
5. Under what conditions on $\mathbf{a}$ and $\mathbf{b}$ is the triangle inequality an equality?

6. Show that $\big| |\mathbf{a}| - |\mathbf{b}| \big| \le |\mathbf{a} - \mathbf{b}|$ for every $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.
7. Suppose $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.
   a. Show that if $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.
   b. Show that if $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$ for some nonzero $\mathbf{c} \in \mathbb{R}^3$, then $\mathbf{a} = \mathbf{b}$.
   c. Show that $(\mathbf{a} \times \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\mathbf{a} \times \mathbf{b})$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are proportional (i.e., one is a scalar multiple of the other).
8. Show that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is the determinant of the matrix whose rows are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ (if these vectors are considered as row vectors) or the matrix whose columns are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ (if they are considered as column vectors).

1.2 Subsets of Euclidean Space

In this section we introduce some standard terminology for sets in $\mathbb{R}^n$.

First, the set of all points whose distance from a fixed point $\mathbf{a}$ is equal to some number r is called the **sphere** of radius r about $\mathbf{a}$, and the set of points whose distance from $\mathbf{a}$ is less than r is called the (open) **ball** of radius r about $\mathbf{a}$. (In ordinary English the word "sphere" is often used for both these purposes, but mathematicians have found it helpful to reserve the word "sphere" for the spherical *surface* and to use "ball" to denote the solid body.) We shall use the notation $B(r, \mathbf{a})$ for the ball of radius r about $\mathbf{a}$:

$$B(r, \mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}| < r\}.$$

Of course, when in dimension 1, a ball is just an open interval, and in dimension 2, the words "disc" and "circle" may be used in place of "ball" and "sphere."

A set $S \subset \mathbb{R}^n$ is called **bounded** if it is contained in some ball about the origin, that is, if there is a constant C such that $|\mathbf{x}| < C$ for every $\mathbf{x} \in S$.

When one studies functions of a single variable, one frequently considers *intervals* in the real line, and it is often necessary to distinguish between *open* intervals (with the endpoints excluded) and *closed* intervals (with the endpoints included). When $n > 1$, there is a much greater variety of interesting subsets of $\mathbb{R}^n$ to be considered, but the notions of "open" and "closed" are still fundamental. Here are the definitions.

Let S be a subset of $\mathbb{R}^n$.

- The **complement** of S is the set of all points in $\mathbb{R}^n$ that are *not* in S; we denote it by $\mathbb{R}^n \setminus S$ or by S^c:

$$S^c = \mathbb{R}^n \setminus S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \notin S\}.$$