Fundamental Theorem of Algebra

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The fundamental theorem of algebra was stated in various forms going back even before Euler. A variety of proofs were proposed. Gauss gave several proofs, not all of them correct. I have not read any of his proofs, nor have I read the proof of Jean-Robert Argand. It is claimed that Argand gave the first correct proof in 1814. I've not been able to read his exact proof, but from what I know of it the proof I am going to give here is pretty close to it. It is totally elementary except for using the fact that a continuous function on a compact set assumes a minimum. Argand must have implicitly assumed that, since he didn't have a construction of the real numbers (for example the least upper bound axiom).

Theorem 1 (Fundamental Theorem of Algebra)**.** *Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where z is a complex variable and the a_j are complex numbers. Then there is a complex number $a \in \mathbb{C}$ for which $f(a) = 0$.*

Proof. Assume not. Then $f(z) \neq 0$ for all $z \in \mathbb{C}$. Consider $|f|$ on the set $K = \{z : |f(z) \leq |f(0)|$. Since $|f(z)| \to \infty$ as $|z| \to \infty$, K is a compact set (closed and bounded). Since K is compact, $|f|$ assumes a minimum on K and this will be a global minimum of $|f|$ on all of $\mathbb{C}$. If the minimum is assumed at a, we can change variables and let $p(z) = f(z + a)$ to get a new polynomial with a minimum at 0. We can multiply p by a constant to arrange it so the minimum value is 1. Let's call the new polynomial q. Then here is how q looks

$$q(z) = 1 + b_k z^k + \cdots + b_n z^n,$$

where the minimum value of $|q|$ is 1 and it is assumed at 0, where in this expression it is assumed that $b_k \neq 0$. This is all done to make the algebra transparent. Now we compare the terms after $b_k z^k$ to $b_k z^k$.

Claim 1. *There is an $\epsilon > 0$ so that*

$$|b_k z^k| \geq 2(|b_{k+1} z^{k+1}| + \cdots + |b_n z^n|), \text{ when } |z| \leq \epsilon.$$

Proof of claim. We can divide by $|z|$ and the claim is equivalent to

$$|b_k| \geq 2(|b_{k+1} z^k| + \cdots + |b_n z^{n-1}|), \text{ when } |z| \leq \epsilon,$$

which is true since the right side goes to 0 as $|z| \to 0$. □

Now choose z to be so that $b_k z^k = -\delta$, where $\delta \leq |b_k| \epsilon^k$. Then $|z|^k \leq \frac{\delta}{|b_k|} \leq \epsilon^k$. Hence

$$\delta \geq 2(|b_{k+1} z^k| + \cdots + |b_n z^{n-1}|).$$

Finally choose δ so that $0 < \delta < 1$. Then for this z

$$\begin{aligned} |q(z)| &= |1 - \delta + b_{k+1} z^{k+1} + \cdots + b_n z^n| && (1) \\ &\leq 1 - \delta + \frac{\delta}{2} && (2) \\ &= 1 - \frac{\delta}{2} && (3) \\ &< 1 && (4) \\ &\to\leftarrow && (5) \end{aligned}$$

□