Hyperbolic Functions

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Many of you are unfamiliar with hyperbolic functions. Here is a crash course on hyperbolic functions. Trigonometric functions could be called circular functions since $(\cos t, \sin t)$ is a parameterization of the circle $x^2 + y^2 = 1$. Similarly $(\cosh t, \sinh t)$ is a parameterization of the hyperbola $x^2 - y^2 = 1$ and hence $\sinh t, \cosh$ are referred to as *hyperbolic functions*. The functions $\sinh t, \cosh t$ are defined as follows.

$$\cosh t = \frac{e^t + e^{-t}}{2}$$
$$\sinh t = \frac{e^t - e^{-t}}{2}$$

It follows that

$$\cosh^2 t - \sinh^2 t = 1. \tag{1}$$

It is also easy to see that

$$\cosh(s+t) = \cosh(s)\cosh(t) + \sinh(s)\sinh(t), \tag{2}$$
$$\cosh(2t) = \cosh^2(t) + \sinh^2(t) \tag{3}$$
$$= 2\cosh^2(t) - 1, \tag{4}$$
$$\sinh(s+t) = \sinh(s)\cosh(t) + \sinh(t)\cosh(s), \tag{5}$$
$$\sinh(2t) = 2\sinh(t)\cosh(t). \tag{6}$$

Also

$$\frac{d}{dt}\cosh t = \sinh t, \tag{7}$$
$$\frac{d}{dt}\sinh t = \cosh t. \tag{8}$$

These functions can come in handy in integration problems. For example let us find an antiderivative of $\sqrt{1+x^2}$. We substitute $x = \sinh t$ to get

$$\begin{aligned}\int \sqrt{1+x^2}\ dx &= \int \cosh^2(t)\ dt \\ &= \int \frac{\cosh(2t)+1}{2}\ dt \\ &= \frac{t}{2} + \frac{\sinh(2t)}{4} \\ &= \frac{t + \sinh(t)\cosh(t)}{2} \\ &= \frac{x\sqrt{1+x^2} + \sinh^{-1}(x)}{2}.\end{aligned}$$

By using the quadratic formula we see that

$$\sinh^{-1}(x) = \log(x + \sqrt{1 + x^2}),$$

and hence

$$\int \sqrt{1 + x^2}\; dx = \frac{x\sqrt{1 + x^2} + \log(x + \sqrt{1 + x^2})}{2}. \tag{9}$$