Math 334 Sample Problems

One notebook-sized page (one side) of notes will be allowed on the test. You may work together on the sample problems – I encourage you to do that. You may ask for help if you bring your work in and show it to us, and tell us where you are stuck. The test (on Monday, October 22) will cover up to and including section 2.1 in the text.

There will be homework problems or example problems from the text on the midterm. There will be four problems on the midterm, at least one will be from this list or a slight modification of a problem on this list. This is a long list of problems that I have accumulated. Do your best to work as many of them as you can. On the midterm you may quote any theorems from the text or results that I stated in class without proof but you may not cite example problems or homework problems.

1. Let the sequence x_n be defined recursively by

$$x_0 = 0, x_{n+1} = \sqrt{3 + x_n}.$$

   Prove that the sequence conveges and find its limit.

2. Let $0 < a < b$. Define two sequences a_n, b_n by the following recursion

$$\begin{aligned} a_1 &= a, \\ b_1 &= b, \\ a_{n+1} &= \sqrt{a_n b_n}, \\ b_{n+1} &= \frac{a_n + b_n}{2}. \end{aligned}$$

   Prove that the limits of the two sequences exist and are the same.

3. Suppose K_m is a sequence of compact connected subsets of $\mathbb{R}^n$ such that $K_{m+1} \subset K_m$. Prove that $C = \cap_{m=1}^{\infty} K_m$ is connected.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function that satisfies:

$$\begin{aligned} f(0) &= 0 \\ f(x) &> 0 \text{ if } x \neq 0 \\ f(x + y) &\leq f(x) + f(y). \end{aligned}$$

   Suppose f is continous at 0.

   (a) Prove that f is continuous everywhere.

   (b) Prove that there is a number $c > 0$ so that $f(x) \geq c$ on $\|x\|_2 = 1$.

5. Let $A_j = \{(1,0),(0,0)\} \cup \{(x,y) : 0 < y < 1/j\}$.
   (a) Prove that A_j is connected and $A_{j+1} \subset A_j$.
   (b) Prove that $\cap_{j=1}^{\infty} A_j$ is not connected.

6. Prove that the set $\{(x, y, \sin(x^2+y^2)) : x^2+y^2 = 1\}$ is a connected subset of $\mathbb{R}^3$. You may use the fact that $\{(x,y) : x^2+y^2 = 1\}$ is connected.

7. Let $c > 0$. Define $g(x) = (\frac{1}{2})(x + \frac{c}{x})$ when $x \neq 0$. Let $x_0 > 0, (x_0)^2 > c$ and define a sequence recursively by $x_{n+1} = g(x_n)$. Prove that $x_n \to \sqrt{c}$.

8. (a) Suppose that $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are bounded and uniformly continous on some set S. then prove that fg is uniformly continuous.
   (b) Prove that x and $\sin(x)$ are uniformly continuous on $\mathbb{R}$ but $x\sin(x)$ is not uniformly continuous on $\mathbb{R}$.

9. Let S be a nonempty set of rational numbers. Suppose that S is connected. Prove that S consists of a single number.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (1 - \cos \frac{x^2}{y})\sqrt{x^2+y^2} \text{ if } y \neq 0 \\ 0 \text{ if } y = 0. \end{cases}$$

    Prove that f is continuous at $(0,0)$.

11. (a) Let f be a continuous function from $[0,1]$to $\mathbb{R}^n$. Prove that the graph of f, $\Gamma_f = \{(x,y) : y = f(x)\}$ is closed. (Γ_f is a subset of $[0,1] \times \mathbb{R}^n$.)
    (b) Let f be a function from $[0,1]$ to $\mathbb{R}^n$. Suppose Γ_f is closed. Is f continuous? Give a proof or counterexample.

12. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be compact. Prove that $A \times B \subset \mathbb{R}^{n+m}$ is compact.

13. Let $\mathbb{Q}$ be the set of rational numbers. Prove that there does NOT exist a pair of non-empty sets A, B such that $\mathbb{Q} = A \cup B$ with $\overline{A} \cap \overline{B} = \emptyset$.

14. Suppose that $f : \mathbb{R} \to \mathbb{R}$, defined on all, of $\mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x-y|^2,$$

    for all $x, y \in \mathbb{R}$. Prove that f is constant.

15. Prove that the temperature of a tetrahedron must have at least three distinct points on the edges or vertices of the tetrahedron with the same value. Assume the temperature is a continuous function. (Hint: the intermediate value theorem.)

16. Prove that the function $\cos(x^2)$ defined on all of $\mathbb{R}$ is *not* uniformly continuous.

17. Let S be an open set and let $p \in S, q \notin S$. Prove that there is a boundary point of S on the line segment joining p and q.

18. Let a sequence be defined recursively by the rules: $x_0 = 1, x_{n+1} = x_n + \dfrac{1}{x_n}$. Prove that the sequence does not converge.

19. Suppose $\{a_n\}$ is a sequence with $a_n > 0$ and b_n is defined by $b_n = a_n + \dfrac{1}{a_n}$.
    (a) Assume that $a_n \geq 1$ and that $\{b_n\}$ converges. Prove that $\{a_n\}$ converges.
    (b) If it is assumed that $\{b_n\}$ converges but only that $a_n > 0$, it does not necessarily follow that $\{a_n\}$ converges. Find such an example.

20. Prove that $\lim\limits_{n\to\infty} \sin n$ does not exist.

21. Prove that the set $\{(x, y, z) : \dfrac{x^2}{2} + \dfrac{y^2}{3} + \dfrac{z^2}{4} = 1\}$ is connected.

22. Suppose A is a connected set and that $A \subset B \subset \overline{A}$. Prove that B is connected.

23. Let f be continuous on $(0, 1)$ and suppose $0 < f(x) < x$ for all $x \in (0, 1)$. Define $f_n(x)$ inductively by $f_1(x) = f(x), f_{n+1}(x) = f(f_n(x))$. Prove that $\lim\limits_{n\to\infty} f_n(x)$ exists and compute it.

24. Let $|a| < 1$, where a is a real number. Prove that $\lim\limits_{n\to\infty} na^n = 0$. Notice that a is allowed to be negative.

25. Let f be a continuous real valued function defined on $[0, 1]$ such that $f(0) = f(1)$. Show that there is a pair of points $a, b \in [0, 1]$ such that $b - a = 1/2$ and $f(b) = f(a)$.

26. You will need to know the the following items
    (a) Cauchy's inequality
    (b) Triangle inequality
    (c) Open set
    (d) Closed set

(e) Boundary and interior of a set
(f) Closure of a set
(g) Compact set
(h) Bolzano-Weierstrass theorem
(i) Connected set
(j) Convergent sequence
(k) Completeness
(l) Cauchy's criterion
(m) Continuity at a point
(n) Continuity on a set
(o) Uniform continuity
(p) Differentiability
(q) Mean Value Theorem
(r) L'Hôpital's Rule