Riemann Integral

November 19, 2011

This note gives a proof that a bounded function is Riemann integrable if and only if it is continuous except on a set of Lebesgue measue 0. We will say f is continuous *almost everywhere* if it is continous except on a set of measure 0. To prove this we will introduce several key ideas.

Let K be a set in $\mathbb{R}^n$.

Definition 1. *The Lebesgue number of an open cover $\{U_\alpha\}$ of a set K is a number $\delta > 0$ with the property that for each point $a \in K$ the set $\{x : |x - a| < \delta\}$ is a subset of some set U_α in the cover.*

Theorem 1. *Every open cover of a compact set has a Lebesgue number.*

Proof. For each point $x \in K$, $x \in U_\alpha$ for some set U_α. Since U_α is open, there is a δ_x so that $\{y : |y - x| < \delta_x\} \subset U_\alpha$. Since K is compact, there is a finite cover $\{W_j\}$ where $W_j = \{y : |y - x_j| < \frac{1}{2}\delta_{x_j}\}$. Let $\delta = \frac{1}{2}\min\{\delta_{x_j}\}$. Let $a \in K$ and let $|y - a| < \delta$. Now $a \in W_{x_j}$ for some j hence $|a - x_j| < \frac{1}{2}\delta_{x_j}$. Then

$$|y - x_j| \leq |y - a| + |a - x_j| < \delta + \frac{1}{2}\delta_{x_j} \leq \delta_{x_j},$$

so $y \in U_\alpha$ for some α. □

(This is the proof that Cory suggested.)

Definition 2. *The **oscillation** of a function on a set S is*

$$\Omega_f(S) = \sup\{f(x) : x \in S\} - \inf\{f(x) : x \in S\}.$$

*The **oscillation function** is*

$$\omega_f(x) = \lim_{\epsilon \to 0} \Omega_f(\{y : |y - x| < \epsilon\}).$$

Remark 1. *f is continuous at x if and only if $\omega_f(x) = 0$.*

Remark 2. *If $\omega_f(x) < \alpha$ then there is neighborhood W of x so that $\Omega_f(W) < \alpha$.*

Proposition 1. *Let f be defined on a compact set. Let $D_\alpha = \{x : \omega_f(x) \geq \alpha\}$. Then D_α is a closed compact set.*

Proof. Let $x \notin D_\alpha$. Then $\omega_f(x) < \alpha$ and hence $\Omega_f(\{y : |y - x| < \epsilon\} < \alpha$ for small enough ϵ. But this implies that $\omega_f(y) < \alpha$ when $|y - x| < \epsilon$ so the complement of D_α is open and D_α is closed. □

Corollary 1. *If A is compact and $\mu(A) = 0$ then $c(A) = 0$.*

Proof. If we have a countable open cover U_k of A such that $\sum |U_k| < \epsilon$ then any finite subcover satisfies $\sum |U_{k_j}| < \epsilon$. □

Let $D_f = \{x : \omega_f(x) > 0$ (the discontinuity set of f).

Theorem 2. *Let f be defined and bounded on an interval $[a, b]$. Then f is Riemann integrable if and only if $\mu(D_f) = 0$ where μ is Lebesgue measure.*

Proof. Assume f is Riemann integrable. We will show $\mu(D_f) = 0$ by showing that $c(D_\alpha) = 0$ for any $\alpha > 0$ where c is the Jordan content of D_α. Suppose

$$S_P(f) - s_P(f) < \alpha\epsilon.$$

Let J_k be the intervals in the partition P that have a point of D_α in their interior. Then

$$\alpha \sum_k |J_k| \leq \sum_k (M_k - m_k)|J_k| < \alpha\epsilon,$$

hence $\sum_k |J_k| < \epsilon$. The intervals J_k cover D_α except for the finite number of points of D_α that do not belong to the interior of some interval of P. We can find a finite number of small additional intervals around these points such that the sum of their lengths is less than ϵ. So we have a finite set of intervals that cover D_α such that the sum of their lengths is less than 2ϵ and hence $c(D_\alpha) = 0$.

Assume that $\mu(D_f) = 0$. Then $\mu(D_\epsilon) = 0$ for all ϵ and since D_ϵ is compact $c(D_\epsilon) = 0$. Choose a finite set of intervals J_k such that $D_\epsilon \subset \cup interior(J_k)$ and $\sum |J_k| < \epsilon$. Notice I chose the same ϵ, which I am allowed to do. Let $K = [a, b] - \cup interior(J_k)$. Then K is a finite union of closed intervals and hence compact. For each $x \in K$ there is an open interval W_x such that $\Omega_f(W_x) < \epsilon$ by definition of $\omega_f(x)$, remark 2, and since $K \cap D_\epsilon = \emptyset$. Let δ be a Lebesgue number for the cover $\{W_x\}$. Choose a (finite) refinement of the intervals in K so that each of the intervals I in the refinement has length less than δ. Then on each of these intervals $\Omega_f(I) = (M_i - m_i) < \epsilon$. So for the partition P consisting of the J_k intervals and the intervals refining K we have

$$S_P(f) - s_P(f) \leq 2M\epsilon + \epsilon(b - a),$$

where we have made an obvious over estimate of the lengths of the intervals in K and used the assumption that $|f(x)| < M$ (f is bounded). Now we are done.

□