Arithmetic mean – Geometric mean

Note Title 11/10/2009

Let $p_1, p_2, \cdots, p_n > 1$. Let $x_1, x_2, \cdots, x_n > 0$.

Suppose $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$. Then

$$\text{(AG)} \qquad x_1^{1/p_1} \cdot x_2^{1/p_2} \cdot \cdots x_n^{1/p_n} \le \frac{1}{p_1} x_1 + \frac{1}{p_2} x_2 + \cdots + \frac{1}{p_n} x_n$$

<u>Proof</u>: Let
$$f(x) = x_1^{1/p_1} \cdot x_2^{1/p_2} \cdot \cdots x_n^{1/p_n}$$

What is the maximum of f on $\frac{1}{p_1} x_1 + \cdots + \frac{1}{p_n} x_n = C$

$x_1, \cdots, x_n \ge 0$? It occurs at the critical pt

determined by:
$$f_{x_1} = \frac{1}{p_1} x_1^{1/p_1 - 1} \cdot x_2^{1/p_2} \cdots x_n^{1/p_n} = \lambda \frac{1}{p_1}$$
$$f_{x_2} = \frac{1}{p_2} x_1^{1/p_1} \cdot x_2^{1/p_2 - 1} \cdot x_3^{1/p_3} \cdots x_n^{1/p_n} = \lambda \frac{1}{p_2}$$
$$\vdots$$
$$f_{x_n} = \cdots = \lambda \frac{1}{p_n}$$

The firs two equations give $x_1^{1/p_1 - 1} x_2^{1/p_2} = x_1^{1/p_1} \cdot x_2^{1/p_2 - 1}$

Hence $x_1 = x_2$. Similarly $x_1 = x_2 = x_3 = \cdots = x_n$

Hence $x_1 = x_2 = \cdots = x_n = C$ is the critical pt

so $x_1^{1/p_1} \cdots x_n^{1/p_n} \le C = \frac{1}{p_1} x_1 + \cdots + \frac{1}{p_n} x_n$ for all other

values of $x_1, \cdots, x_n$. This is true for all c, so (AG)

is proved.

We will now apply (AG) to prove Holder's inequality.

(H) Let $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ "conjugate exponents".

Then $$\left| \sum x_i y_i \right| \leq \left(\sum |x_i|^p\right)^{1/p} \left(\sum |y_i|^q\right)^{1/q}$$

We'll abbreviate this using the notation:

$$\langle x, y \rangle = \sum x_i y_i, \quad \|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$$

as

(H) $$|\langle x, y \rangle| \leq \|x\|_p \cdot \|y\|_q$$

Proof (H) is true if either $x = 0$ or $y = 0$.

Suppose $x \neq 0$ and $y \neq 0$. The inequality can be re-written as $$\left| \left\langle \frac{x}{\|x\|_p}, \frac{y}{\|y\|_q} \right\rangle \right| \leq 1.$$

So we are reduced to proving $|\langle u, v \rangle| \leq 1$, if $\|u\|_p \leq 1$, $\|v\|_q \leq 1$. Now use (AG) in the following form:

$$|u_i||v_i| \leq \frac{1}{p}|u_i|^p + \frac{1}{q}|v_i|^q,$$

sum and get

$$\left|\sum u_i v_i\right| \leq \sum |u_i||v_i| \leq \frac{1}{p}\sum |u_i|^p + \frac{1}{q}\sum |v_i|^q$$
$$= \frac{1}{p} + \frac{1}{q} = 1.$$

Next we apply Holder's inequality (H) to get Minkowski's inequality

Let $p > 1$. Then

(M) $$\|x+y\|_p \le \|x\|_p + \|y\|_p$$

Proof:

$$|x_i+y_i|^p = |x_i+y_i| \cdot |x_i+y_i|^{p-1}$$
$$\le (|x_i|+|y_i|)\,|x_i+y_i|^{p-1}$$

sum and use (H) with $\frac{1}{q} = 1-\frac{1}{p} = \frac{p-1}{p}$, $p=(p-1)\cdot q$

$$\|x+y\|_p^p \le \left(\|x\|_p + \|y\|_p\right)\left(\sum |x_i+y_i|^{(p-1)q}\right)^{1/q}$$
$$= \left(\|x\|_p + \|y\|_p\right) \|x+y\|_p^p \cdot \|x+y\|_p^{-1}$$

Cancel $\|x+y\|_p^p$ to get

$$\|x+y\|_p \le \|x\|_p + \|y\|_p .$$

(Notice, we may assume $\|x+y\|_p \ne 0$.)

As a last example we briefly disuss generalized means (sometimes called power means or Holder means). Suppose $x_1, \dots x_n > 0$

$$H_p(x) = H_p(x_1, \dots x_n) = \left(\frac{1}{n}(x_1^p + \dots + x_1^p)\right)^{1/p}, \quad \text{for } p \ne 0.$$

We will assume $p > 0$, but that is not necessary.

Suppose $0 < p < q$. Then

(HI) $$H_p(x) < H_q(x)$$

Proof: Let $t = q/p > 1$, and $y_j = x_j^p$,

$$\frac{1}{s} = 1 - \frac{1}{t} = \frac{t-1}{t} = \frac{q-p}{q} = 1 - p/q, \quad \frac{1}{t} = p/q$$

$$\sum x_j^p = \sum 1 \cdot y_j \le (n)^{1/s} \cdot \left(\sum y_j^t\right)^{1/t}$$

$$= n^{1/s} \cdot \left(\sum x_j^q\right)^{p/q}$$

$$= n \cdot \left(\frac{1}{n} \sum x_j^q\right)^{p/q}.$$

So

$$H_p(x) = \left(\frac{1}{n} \sum x_j^p\right)^{1/p} \le \left(\frac{1}{n} \sum x_j^q\right)^{1/q} = H_q(x)$$

QED.

Remarks.

1. Replace x_j with $\frac{1}{x_j}$ to get

$$\left(\frac{1}{n} \sum x_j^{-p}\right)^{1/p} \le \left(\frac{1}{n} \sum x_j^{-q}\right)^{1/q} \quad \text{and hence}$$

$$= \left(\sum \frac{1}{n} x_j^{-q}\right)^{1/(-q)} \le \left(\frac{1}{n} \sum x_j^{-p}\right)^{1/(-p)}$$

when $-q < -p < 0$.

2. $$\left(\prod x_i\right)^{p/n} = \left(\prod x_i^p\right)^{1/n} \le \frac{1}{n}\left(\sum x_i^p\right), \text{ so}$$

$$\left(\prod x_i\right)^{1/n} \le \left(\frac{1}{n}\left(\sum x_i^p\right)\right)^{1/p} = H_p(x), \quad p > 0$$

Using this we prove:

$$H_p(x) < H_q(x) \text{ if } p < 0 < q$$