Norms

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This note will discuss some norms on $\mathbb{R}^n$ that are most useful in analysis. In Folland's book there is only one norm used and it is denoted by $|x|$. In this note norms will be denoted by double bars, $||x||$.

The Euclidean norm, $||x||_2 = (x_1^2 + x_2^2 + \dots x_n^2)^{1/2}$ is the norm that is commonly used in definitions of such things as continuity and differentiability. Sometimes it is inconvenient and proofs are more easily made using a different norm or a mix of two norms. A *norm* is a way of measuring the size of a vector by assigning to it a non-negative number with certain simple properties. A norm is a special type of *metric* defined on a vector space. The following are the defining properties of a norm.

$$||x|| \geq 0, \text{ and } ||x|| = 0 \text{ implies } x = 0 \tag{1}$$

$$||ax|| = |a|||x||, \text{ where } a \in \mathbb{R} \tag{2}$$

$$||x + y|| \leq ||x|| + ||y||, \text{ triangle inequality} \tag{3}$$

The following are norms we will find useful.

1. $$||x||_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}, \text{ the sup or } \infty\text{-norm}$$

2. $$||x||_1 = |x_1| + |x_2| + \dots + |x_n|, \text{ the 1-norm}$$

3. $$||x||_2 = (x_1^2 + x_2^2 + \dots x_n^2)^{1/2}, \text{ the Euclidean or 2-norm}$$

4. $$||x||_p = (|x_1|^p + |x_2|^p + \dots |x_n|^p)^{1/p}, \ p \geq 1, \text{ the } p\text{-norm}$$

It's easy to verify that the ∞-norm and 1-norm satisfy the triangle inequality. The triangle inequality for the p-norms with $p > 1$ is not trivial. The proof will be given post on arithmetic-geometric mean on the class website.

The norms are all *equivalent* in a way that makes it possible to define continuity and limits in a flexible way. This is the simplest version of that equivalence.

Theorem 1.

$$||x||_1 \geq ||x||_2 \geq ||x||_\infty \geq \frac{1}{n}||x||_1.$$

Proof. Let m be the index for which $|x|_m = ||x||_\infty$. Then

$$\left(\sum_1^n |x_j|\right)^2 \geq \sum_1^n |x_j|^2 \geq |x_m|^2.$$

This proves the first two inequalities. Also

$$n||x||_\infty = n|x_m| \geq \sum_1^n |x_j| = ||x||_1$$

since $|x_j| \leq |x_m|$ for all j. This proves the theorem. □

The next result demonstrates how the norms can be mixed in a statement.

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined in a neighborhood of a point $a \in \mathbb{R}^n$. Suppose we can prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that $||x - a||_1 < \delta$ implies that $||f(x) - f(a)||_\infty < \epsilon$. Then f is continuous at a.*

Proof. If we can make $||f(x)-f(a)||_\infty < \dfrac{\epsilon}{n}$ then it will follow that $||f(x)-f(a)||_2 < \epsilon$ since $||y||_2 \leq ||y||_1 \leq ||y||_2$ for any y. We can make $||f(x) - f(a)||_\infty < \dfrac{\epsilon}{n}$ by choosing $||x - a||_1 < \delta$. Now if $||x = a||_2 < \dfrac{\delta}{n}$ then $||x = a||_1 < \delta$ and so $||x = a||_2 < \dfrac{\delta}{n}$ implies $||f(x) - f(a)||_2 < \epsilon$. □

By the same type of reasoning we can prove the following theorem

Theorem 3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined in a neighborhood of a point $a \in \mathbb{R}^n$. f is continuous at a if for any $\epsilon > 0$ there is a $\delta > 0$ so that if $||x - a||_p < \delta$ then $||f(x) - f(a)||_q < \epsilon$.*