Math 334 Sample Problems

One side of one notebook sized page of notes will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover up to §3.3.

1. Use the method of Lagrange multipliers to find the distance from the point $(0, b)$ to the curve $x^2 = 4y$.
2. Find the maximum of xy^2z^3 on the set $\{(x, y, z) : x \geq 0, y \geq 0, z \geq 0, x + y + z = 12\}$.
3. Suppose that $F(u)$ is a differentiable function with $F'(u) \neq 0$ and $z = f(x, y)$ is a differentiable function that satisfies and
$$z = F(ax + by + cz),$$
where a, b, c are constants with $c \neq 0$. Prove
$$bf_x = af_y.$$
4. Prove that the level set $y^2 = (x - 1)(x - 2)(x - 3)$ is the disjoint union of two smooth connected curves. Prove that one of the curves is compact and the other is not.
5. Suppose that (x_0, y_0, z_0, u_0) satisfies the equations
$$\begin{aligned} x + y + z &= F(u) \\ x^2 + y^2 + z^2 &= G(u) \\ x^3 + y^3 + z^3 &= H(u), \end{aligned}$$
where, F, G, H are C^1 in a neighborhood of u_0. State a sufficient condition for being able to solve these equations for x, y, z as C^1 functions of u in a neighborhood of (x_0, y_0, z_0, u_0).
6. Is the set $\{(x, y) : y^2 + x^2e^y = 0\}$ a smooth curve? Is the set $\{(a \cos t, b \sin t) : t \in (0, \pi)\}$, where $a > 0, b > 0$ a smooth curve?
7. Expand $(1 - x + 2y)^3$ in powers of $x - 1$ and $y - 2$ in two different ways. The first way is by using algebra and the second way is by computing the Taylor polynomial of degree three centered at $(1, 2)$.
8. Using the method of Lagrange multipliers, find the highest and lowest points of the circle
$$x^2 + y^2 + z^2 = 16,\ (x + 1)^2 + (y + 1)^2 + (z + 1)^2 = 27$$

9. Show that the surface $z = 3x^2 - 2xy + 2y^2$ lies entirely above every one of its tangent planes. Hint: Look at the Taylor expansion at every point.

10. Let $a > 0, b > 0$ and $a + b = 1$. Also let $x > 0, y > 0$. prove that
$$x^a y^b \leq ax + by,$$
by using the method of Lagrange multipliers applied to maximize $x^a y^b$ subject to $ax + by = c$, where $c > 0$ is some constant.

11. Let $f(x, y) = x^2(1 + y)^3 + 7y^2$ define a function on $\mathbb{R}^2$. Find and classify its critical points. What is $\sup\{f(x, y) : (x, y) \in \mathbb{R}^2\}$? What is $\inf\{f(x, y) : (x, y) \in \mathbb{R}^2\}$?

12. Define $f(x) = (\log x)^{\log x}$, for $x > 1$. Using the chain rule, compute $f'(x)$.

13. Folland, §2.9, problem 16.

14. Suppose $F(x, y)$ is a C^2 function that satisfies the equations $F(x, y) = F(y, x), F(x, x) = x$. Prove that the quadratic term in the Taylor polynomial of F based at the point (a, a) is $\frac{1}{2}F_{xx}(a, a)(x - y)^2$.

15. Prove that there is a continuously differentiable function $f(x, y)$ defined in a neighborhood of $(0, 1/2)$ such that
$$x + y + f(x, y) - e^{xyf(xy)} = 0,$$
and $f(0, 1/2) = 1/2$.

16. Let f be C^2 on (a, b) and suppose $f''(x) \geq 0$ for all $x \in (a, b)$. Let $x_1, x_2, \ldots, x_n$ be points in (a, b). Prove that
$$f(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)) \leq \frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n)).$$

17. Let $: \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x, y) = \begin{cases} \frac{y^3}{x^2+y^2}, \text{ when } x^2 + y^2 \neq 0 \\ 0, \text{ when } (x, y) = (0, 0). \end{cases}$$
Prove that f is continuous on all of $\mathbb{R}^2$, f is differentiable for $(x, y) \neq (0, 0)$, f_x, f_y exist at all points of $\mathbb{R}^2$, f is not differentiable at $(0, 0)$.

18. Suppose f is C^2 on an open interval in $I \subset \mathbb{R}$ and x_1, x_2, x_3 are distinct points of I. Prove that there exists $y \in I$ such that
$$f(x_1)(x_3 - x_2) - f(x_2)(x_3 - x_1) + f(x_3)(x_2 - x_1) = \frac{1}{2}f''(y)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

19. There may be homework problems or example problems from the text or lectures on the midterm.
20. The following topics have been covered since the first midterm:
    (a) Chain rule.
    (b) Mean value theorem.
    (c) Higher order partials and equality of mixed partials.
    (d) Taylor's theorem in one and several variables with Lagrange's form of the remainder.
    (e) Behavior near critical points – second derivative test for extrema in the case of two variables.
    (f) Max-min problems with constraints. The method of Lagrange multipliers.
    (g) Implicit Function Theorem.
    (h) Smooth curves and surfaces.