Alternating Harmonic Series

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We will compute the sum of a rearranged alternating harmonic series. Sum the series in the following manner: add the first p positive terms then the first n negative terms, then the next p positive terms, then the next n negative terms, ... The sum of the first m such groups looks like

$$\begin{aligned} s_{m(p+n)} = &\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2p-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2p+1} + \cdots + \frac{1}{4p-1}\right) \\ &- \left(\frac{1}{2n+2} + \cdots + \frac{1}{4n}\right) + \cdots + \left(\cdots + \frac{1}{2mp-1}\right) - \left(\cdots + \frac{1}{2mn}\right) \end{aligned}$$

Let

$$\begin{aligned} h_m &= 1 + 1/2 + \cdots + 1/m \\ e_m &= h_m - \log(m). \end{aligned}$$

Then

$$\begin{aligned} s_{m(p+n)} &= h_{2mp-1} - \frac{1}{2}h_{mp-1} - \frac{1}{2}h_{mn} \\ &= e_{2mp-1} - \frac{1}{2}e_{mp-1} - \frac{1}{2}e_{mn} + \log(2mp-1) - \frac{1}{2}\log(mn(mp-1)) \end{aligned}$$

Now $\lim_{n\to\infty} e_n = \gamma$, Euler's constant, so

$$\begin{aligned} \lim_{m\to\infty} s_{m(p+n)} &= \lim_{m\to\infty} \log\left[\frac{2mp-1}{\sqrt{mn(mp-1)}}\right] \\ &= \log 2 + \frac{1}{2}\log\frac{p}{n} \end{aligned}$$

This proves that the subsequence $s_{m(p+n)}$ converges and it is easy to show that the rearranged sum converges to limit of this subsequence. (Estimate $s_j - s_{m(p+n)}$.)