Potential of a Charged Sphere

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This note deals with the value of the potential of a uniformly charged sphere of radius a at a point on the sphere. We take the definition of the potential $u(\mathbf{x})$ at $\mathbf{x}$ to be

$$u(\mathbf{x}) = \sigma \int_{|\mathbf{y}|=a} \frac{d\mathbf{y}}{|\mathbf{y}-\mathbf{x}|}. \tag{1}$$

This is not an improper integral if $r \neq a$, where $r = |\mathbf{x}|$. The value was computed as homework problem number 1 in section 5.6. The value depends only on r and is

$$\begin{cases} 4\pi\sigma a, & a < r, \\ \frac{4\pi\sigma a^2}{r}, & a > r. \end{cases} \tag{2}$$

Notice that this function has a limit as $r \to a$ and the answer depends only r. However that doesn't mean that the improper integral converges at a point $\mathbf{x_0}$ on the sphere and even if it does, it doesn't imply that

$$\lim_{\mathbf{x}\to\mathbf{x_0}} \int_{|\mathbf{y}|=a} \frac{d\mathbf{y}}{|\mathbf{y}-\mathbf{x}|} = \int_{|\mathbf{y}|=a} \frac{d\mathbf{y}}{|\mathbf{y}-\mathbf{x_0}|}. \tag{3}$$

We will compute the improper integral and see that (3) is true. Assume that $\mathbf{x_0} = (0, 0, a)$. Then the improper integral in spherical coordinates is

$$\lim_{\epsilon\to 0} \left(\sigma a^2 \int_0^{2\pi} \int_\epsilon^\pi \frac{\sin\phi d\phi}{\sqrt{2a^2 - 2a^2\cos\phi}} \right) \tag{4}$$

$$\lim_{\epsilon\to 0} \left(\sigma\sqrt{2}a \int_\epsilon^\pi \frac{\sin\phi d\phi}{\sqrt{1-\cos\phi}} \right) \tag{5}$$

Make the substitution $u = 1 - \cos\phi$, compute the integral, and take to limit to get

$$\sigma 4\pi a \tag{6}$$