Euler's Constant

January 30, 2008

This note has some details on Euler's constant, γ. First the basic theorem.

Theorem 1. *Let* $h_m = 1 + 1/2 + 1/3 + \cdots + 1/m$ *be the* m^{th} *partial sum of the harmonic series. Let* $e_m = h_m - \log m$. *Then* $e_m > 0$ *and* $e_{m+1} < e_m$. *Hence* $\gamma = \lim_{m\to\infty} e_m$ *exists.*

Proof. Since $1/x$ is a strictly decreasing positive function, $1 + 1/2 + 1/3 + \cdots + 1/(m-1) > \int_1^m dx/x$. So $e_m > 0$. Also $e_{m+1} = e_m + (1(m+1) - \log[(m+1)/m]$. Again, since $1/x$ is decreasing, $(1(m+1) - \log[(m+1)/m] = 1/(m+1) - \int_m^{m+1} dx/x < 0$. This implies $e_{m+1} < e_m$, so $\gamma = \lim\limits_{m\to\infty} e_m$ exists. □

Next some estimates.

Theorem 2. *Let* $g_m = 1 + 1/2 + 1/3 + \cdots + 1/m - \log(m+1)$. *Then* $g_{m+1} > g_m$ *and* $\lim\limits_{m\to\infty} g_m = \gamma$.

Proof. We prove the last statement first. $e_m = g_m - \log[m/(m+1)]$. Since $\log[m/(m+1)] \to 0$ the last statement is proved. Now as in the argument of Theorem 1, $g_{m+1} = g_m + 1/(m+1) - \log[(m+2)/(m+1)]$, so $g_{m+1} > g_m$. □

Corollary 1.

$$g_5 = 0.491573864105278 < \gamma < 0.673895420899233 = e_5$$

Remark It takes a lot of computing to get much accuracy this way. There are better ways.