Math 335 Sample Problems

One notebook-sized page of notes (both sides may be used) will be allowed on the final exam. The final will be comprehensive.

1. Prove that
$$\lim_{n\to\infty} \int_0^\pi \frac{\sin(nx)}{x} dx = \frac{\pi}{2}.$$

2. Define a function $\log_p(x)$ inductively by the formulas $\log_0(x) = x, \log_{p+1}(x) = \log(\log_p(x))$. Prove by induction that the series
$$\sum_{n=m}^{\infty} \frac{1}{\log_0(n)\log_1(n)\log_2(n)\ldots\log_p(n)}$$
(where m is large enough for the denominators to be defined as real numbers) diverges for every p.

3. Suppose that $a_n > 0$, that a_n is decreasing, and that $\displaystyle\sum_1^\infty a_n$ converges. Is it true that $\lim_{n\to\infty} na_n = 0$? If true prove it, if false give a counterexample.

4. Show that the series $\displaystyle\sum_1^\infty \frac{\sin nx}{\sqrt{n}}$ converges for all x and uniformly on any interval of the form $[\delta, 2\pi - \delta]$, where $\delta > 0$ is small. Show that the series is not the Fourier series of a Riemann integrable function.

5. Find the solution of $u_t = 3u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \cos x \sin 5x$. (This is easier than it looks.)

6. (a) Let $\sum_0^\infty a_n x^n$ be a series with radius of convergence R. Substitute $re^{i\theta}$ for x and get a new series involving $e^{in\theta}$. If $0 < r < R$ prove that this is a Fourier series (the variable is θ).

   (b) Prove that $\sum_0^\infty r^{2n}|a_n|^2$ converges for $0 \le r < R$.

7. Prove that

$$\int_0^1 (1 - t^4)^{-1/2} dt = \frac{\Gamma(\frac{5}{4})\sqrt{\pi}}{\Gamma(\frac{3}{4})}.$$

8. Folland, §8.6: problem 10.

9. You may assume that $\sum_{n\neq 0} \frac{e^{inx}}{n}$ is the Fourier series of a Riemann integrable function. Since $\sum_{n>0} \frac{1}{n^2} < \infty$, the Riesz-Fischer Theorem asserts that $\sum_{n>0} \frac{e^{inx}}{n}$ is also the Fourier series of a function in $L^2[-\pi, \pi]$. Prove that $\sum_{n>0} \frac{e^{inx}}{n}$ is *not* the Fourier series of a piecewise continuous function. (Even more is true: this series is not the Fourier series of any Riemann integrable function.) You may use the fact that $-\log(1-z) = \sum_{n>0} \frac{z^n}{n}$, $|z| < 1$ and that the real part of $\log(1 - z) = \log(|1 - z|)$.

10. Find the function (it's a polynomial of degree 2) represented by the series $\sum_{k\in\mathbf{Z},k\neq 0} \frac{e^{ikx}}{k^2}$ by using the Fourier series for the 2π-periodic function equal to x on $(0, 2\pi)$. You may use $\sum_{k\in\mathbf{Z},k\neq 0} \frac{1}{k^2} = \frac{\pi^2}{3}$.

11. Let f be a 2π-periodic function and let a be a fixed real number and let a new function g be defined by $g(x) = f(x-a)$. What is the relation between the Fourier coefficients $\widehat{f}(n)$ and $\widehat{g}(n)$?

12. Let f be a 2π-periodic, piecewise smooth function. Let $\widehat{f}(n)$ be the complex Fourier coefficients of f. Show that there is a constant M (which will depend on f) such that $|\widehat{f}(n)| < M/n$ for all n. Do **not** assume f is continuous.

13. Let f and g be continuous 2π-periodic functions. Define the *convolution* of f and g to be the function. $f * g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)g(t)dt$.
    (a) Prove that $f * g$ is 2π-periodic.
    (b) Prove that $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$, so the Fourier series of $f * g$ is $\sum_{-\infty}^{\infty} c_n d_n e^{inx}$, where $c_n = \widehat{f}(n),\ d_n = \widehat{g}(n)$.

14. (a) Find the cosine series of f where
    $f(x) = 0, 0 < x < \pi/2;\ f(x) = 1, \pi/2 < x < \pi$.
    (b) Prove that the series converges for all x.
    (c) For which x does the series converge absolutely?

15. Find the Fourier series of
$$\frac{1-r^2}{1-2r\cos x + r^2}$$
    where $0 \le r < 1$. (You don't need to integrate.)

16. let f be 2π-periodic, continuous, and piecewise smooth. Let m be any positive integer and define the function f_m by the formula $f_m(x) = f(mx)$. Prove that $\widehat{f_m}(n) = \widehat{f}\left(\frac{n}{m}\right)$ if m divides n and is 0 otherwise.

17. Determine a, b, c so that $f_0(x) = 1, f_1(x) = x + a, f_2(x) = x^2 + bx + c$ is an orthogonal set using the inner product $< f, g >= \displaystyle\int_0^2 fg$ on $[0, 2]$.

18. There may be problems from the text, statements of theorems from the text, problems from previous review sets, or examples from class on the exam.