# CAUCHY-BINET

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**Theorem 0.1.** *(Cauchy-Binet) Let $A$ be a $k \times n$ matrix and $B$ be an $n \times k$ matrix. Then*

$$\det(AB) = \sum_J \det(A(J))\det(B(J)),$$

*where $J = (j_1, j_2, \ldots, j_k)$, $1 \le j_1 < j_2 < \cdots < j_k \le n$, runs through all such multi-indices, $A(J)$ denotes the matrix formed from $A$ using columns $J$ (in that order), and $B(J)$ denotes the matrix formed using rows $J$ of $B$ in that order.*

*Proof.* By definition of matrix product

$$\det AB = \det \begin{bmatrix} \sum_{j_1=1}^n a_{1j_1}b_{j_1 1} & \ldots & \sum_{j_k=1}^n a_{1j_k}b_{j_k k} \\ \vdots & \vdots & \vdots \\ \sum_{j_1=1}^n a_{kj_1}b_{j_1 1} & \ldots & \sum_{j_k=1}^n a_{kj_k}b_{j_k k} \end{bmatrix}$$
$$= \sum_{j_1,\ldots,j_k=1}^{n} \det(A(j_1, j_2, \ldots, j_k) b_{j_1 1} b_{j_2 2} \ldots b_{j_k k},$$

by the multi-linearity of the determinant. Since $\det(A(j_1, j_2, \ldots, j_k) = 0$ if the indices $j_i$ are not all distinct, only those sets of indices occur in the sum. For a fixed multi-index $J' = (j_1', j_2', \ldots, j_k')$ with $1 \le j_1' < j_2' < \cdots < j_k' \le n$ and $J$ some multi-index with these indices in some order, let $j_i' = j_{\sigma(i)}$ where $\sigma$ is a permutation of $[k]$. Then

$$\det(A(j_1, j_2, \ldots, j_k) = \operatorname{sgn}(\sigma)\det(A(j_1', j_2', \ldots, j_k').$$

Now let $J'$ be fixed, and sum over all $J$ which are permutations of $J'$. Let $\tau$ be the inverse of $\sigma$. Then $j_i = j_{\sigma\tau(i)} = j'_{\tau(i)}$. So the sum multiplying $\det(A(j_1', j_2', \ldots, j_k') = \det(A(J'))$ is

$$\sum_\sigma \operatorname{sgn}(\sigma) b_{j'_{\tau(1)}1} b_{j'_{\tau(2)}2} \ldots b_{j'_{\tau(k)}k}$$
$$= \sum_\tau \operatorname{sgn}(\tau) b_{j'_{\tau(1)}1} b_{j'_{\tau(2)}2} \ldots b_{j'_{\tau(k)}k}$$
$$= \det B(J').$$

Hence

$$\det(AB) = \sum_{J'} \det(A(J'))\det(B(J')).$$

□

**Corollary 0.1.**

$$\det AA^T = \sum_J (\det A(J))^2.$$

Here's an application.

**Corollary 0.2.** *Let $\Pi$ be a k-parallelepiped in $\mathbb{R}^n$ and let $\Pi_J$ be the orthogonal projection of $\Pi$ onto the k-dimensional subspace spanned by the $x_J$ axes. Let $m_J = \mu(\Pi_J)$ be the k-dimensional measure of this k-parallelepiped. Then*

$$(\mu(\Pi)^2 = \sum_J m_J^2 = \sum_J \mu(\Pi_J)^2.$$

*Proof.* Recall that if $v_1, v_2, \ldots, v_k$ are the $k$ row vectors which are the spanning edges of $\Pi$ and $V$ is the $k \times n$ matrix defined by

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix},$$

then the measure of $\Pi$ is $\sqrt{\det VV^T}$. Now use the previous corollary and interpret each term in the sum as the square of the measure of a k-parallelepiped in $\mathbb{R}^k$. □

This is a sort of Pythagorean theorem, generalizing the length (1-dimensional measure) formula for a line segment in $\mathbb{R}^n$.

**Application 0.1.** *Let the surface $S$ in $\mathbb{R}^4$ be defined by the parameterization $(x, y) \to (x, y, f(x, y), g(x, y)), (x, y) \in D \subset \mathbb{R}^2$. Then the area of $S$ is*

$$\int_D \left( 1 + f_x^2 + f_y^2 + g_x^2 + g_y^2 + \left( \frac{\partial(f,g)}{\partial(x,y))} \right)^2 \right)^{1/2} dxdy.$$

*The general result for this type of parameterization is as follows. Let $(x_1, x_2, \ldots, x_k) = x_K \in D \subset \mathbb{R}^k$ and let $(f_1(x_K), \ldots, f_m(x_K)) = f_M(x_K)$ be an $M$-tuple of differentiable functions defined on $D$. Let $\mathcal{M} = \{(x_K, f_M(x_K) : x_K \in D\}$. Then the k-dimensional measure of $\mathcal{M}$ is*

$$\int_D \left( 1 + \sum_{I,J,1 \le |I| = |J| \le k} \left( \frac{\partial f_I}{\partial x_J} \right)^2 \right)^{1/2} dx_K.$$