Math 335 Sample Problems

One notebook sized page of notes (*one side*)will be allowed on the test. You may work together on the sample problems – I encourage you to do that. The test will cover 4.5-4.7, 5.4-5.8, and 6.1-6.4. The midterm is on Monday, February 3.

1. Let C be the curve of intersection of $y + z = 0$ and $x^2 + y^2 = a^2$ oriented in the counterclockwise direction when viewed from a point high on the z-axis. Use Stokes' theorem to compute the value of $\int_C (xz + 1)dx + (yz + 2x)dy$.

2. (a) Prove that $\int_C \frac{-ydx + xdy}{x^2 + y^2}$ is not independent of path on $\mathbf{R}^2 - \mathbf{0}$.

   (b) Prove that $\int_C \frac{xdx + ydy}{x^2 + y^2}$ is independent of path on $\mathbf{R}^2 - \mathbf{0}$. Find a function $f(x, y)$ on $\mathbf{R}^2 - \mathbf{0}$ so that $\nabla f = (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2})$.

3. Assume $a_n \geq 0$ for all $n \geq 1$. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges. Give an example of a sequence $a_n \geq 0$ such that $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges and $\sum_1^\infty a_n$ diverges.

4. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \frac{\sqrt{a_n}}{n}$ converges. (Assume $a_n \geq 0$.)

5. Let x_n be a convergent sequence and let $c = \lim_{n \to \infty} x_n$. Let p be a fixed positive integer and let $a_n = x_n - x_{n+p}$. Prove that $\sum a_n$ converges and
$$\sum_1^\infty a_n = x_1 + x_2 + \ldots x_p - pc.$$

6. Suppose $a_n > 0,\ b_n > 0$ for all $n > 1$. Suppose that $\sum_1^\infty b_n$ converges and that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Prove that $\sum_1^\infty a_n$ converges.

7. Let S be the set of all positive integers whose decimal representation does *not* contain 2. Prove that $\sum_{n\in S} \frac{1}{n}$ converges.

8. Prove that $\int_0^\infty \cos x^2 dx$ converges, but not absolutely.

9. Let $a = \lim_{n\to\infty} a_n$. Prove that $\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = a$.

10. Decide if the following integrals converge conditionally, converge absolutely, or diverge.

    (a)
$$\int_{-\infty}^{+\infty} x^2 e^{-|x|} dx$$

    (b)
$$\int_0^\pi \frac{dx}{(\cos x)^{\frac{2}{3}}}$$

    (c)
$$\int_1^\infty \frac{\sin(1/x)}{x} dx$$

11. Let f and g be integrable on $[a, b]$ for every $b > a$.

    (a) Prove that
$$(\int_a^b |fg|)^2 \leq \int_a^b f^2 \int_a^b g^2.$$

    (b) Prove that if $\int_a^\infty f^2$ and $\int_a^\infty g^2$ converge then $\int_a^\infty fg$ converges absolutely.

12. (a) Suppose $\sum_1^\infty a_n$ converges. Fix $p \in \mathbb{Z}^+$. Prove that $\lim_{n\to\infty}(a_n + a_{n+1} + \ldots a_{n+p}) = 0$.

    (b) Suppose $\lim_{n\to\infty}(a_n + a_{n+1} + \ldots a_{n+p}) = 0$ for every p. Does $\sum_1^\infty a_n$ converge?

13. Let $a_n > 0$ and suppose $a_n \geq a_{n+1}$. Prove that $\sum_1^\infty a_n$ converges if and only if $\sum_0^\infty a_{3^n}$ converges.

14. Let S be the surface (torus) obtained by rotating the circle $(x-2)^2 + z^2 = 1$ around the z-axis. Compute the integral $\int_S \mathbf{F} \cdot \mathbf{n} dA$, where $\mathbf{F} = (x + \sin(yz), y + e^{x+z}, z - x^2 \cos y)$.

15. Let $w(x)$ satisfy $w''(x) + w(x) = 0, w(0) = 0, w'(0) = 1$. Let $f(x) = \int_0^x (w(x-y))h(y)dy$. Prove that

$$f''(x) + f(x) = h(x), f(0) = 0, f'(0) = 0.$$

16. Suppose $a_n > b_n > 0$, $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$. Does $\sum_1^\infty (-1)^n b_n$ converge? Give a proof or a counterexample.

17. We have covered the following:

    (a) Divergence theorem
    (b) Stokes' theorem
    (c) Integrating vector derivatives
    (d) Integrals dependent on a parameter
    (e) Improper single and multiple integrals
    (f) Convergence and divergence of a series
    (g) Comparison test
    (h) Integral test
    (i) Cauchy condensation test
    (j) Root test and ratio test
    (k) Absolute and conditional convergence of a series
    (l) Dirichlet's test
    (m) Abel's test and theorem

18. There may be homework problems or example problems from the text on the midterm.