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ARZELÀ'S DOMINATED CONVERGENCE THEOREM FOR THE RIEMANN INTEGRAL

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1. Introduction. Riemann's definition ([14], p. 239) of a definite integral gave rise to a number of important developments in analysis. In the course of these developments a remarkable result due to C. Arzelà ([1], 1885) marked the beginning of a deeper understanding of the continuity properties of the Riemann integral as a function of its integrand. The result of Arzelà we have in mind is the so-called ARZELÀ DOMINATED CONVERGENCE THEOREM for the Riemann integral concerning the passage of the limit under the integral sign. It reads as follows.

THEOREM A (C. Arzelà, 1885). *Let* $\{f_n\}$ *be a sequence of Riemann-integrable functions defined on a bounded and closed interval* $[a, b]$, *which converges on* $[a, b]$ *to a Riemann-integrable function* f. *If there exists a constant* $M>0$ *satisfying* $|f_n(x)| \leqq M$ *for all* $x \in [a, b]$ *and for all* n, *then* $\lim_{n\to\infty} \int_a^b |f_n(x)-f(x)|\,dx=0$. *In particular,*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

Usually, Arzelà's theorem is formulated as a result about term-by-term

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integration for infinite series of integrable functions. In that case, the sequence of the partial sums of the infinite series plays the role of the sequence $\{f_n\}$ in Theorem A.

Due to the development of the theory of Lebesgue integration we recognize nowadays that Arzelà's theorem rests on the countable additivity property of the Lebesgue measure. As a matter of fact, Arzelà based the proof of his theorem on the following result about systems of intervals.

THEOREM B (C. Arzelà, 1885). *Assume for each n that D_n denotes a subset of $[a, b]$ that is the union of a finite (or countably infinite) number of mutually disjoint intervals. If for each n, the sum $\ell(D_n)$ of the lengths of the intervals in each D_n satisfies $\ell(D_n) > \delta$, where $\delta > 0$, then there exists at least one point $c \in [a, b]$ which satisfies $c \in D_n$ for infinitely many n.*

If one knows that the interval function measuring the length of an interval is countably additive, then Arzelà's Theorem B follows immediately by observing that

$$\ell(\limsup_{n\to\infty} D_n) \geqq \limsup_{n\to\infty} \ell(D_n) \geqq \delta > 0.$$

In the axiomatic approach to the theory of integration, Arzelà's theorem in one form or another is taken as one of the basic axioms. For instance, in abstract measure theory (see [18], Chap. 2) it appears in the form of the axiom that measures are countably additive. One of the basic axioms introduced by Daniell (see [7], [3] and [18], Chap. 3), in his functional approach to the theory of integration, is Arzelà's theorem for decreasing sequences of functions that decrease everywhere to zero. The extension procedures of the theory of integration for abstract measures as well as for abstract integrals are such that all the axioms are preserved unter the extension. Consequently, the general dominated convergence theorem for the extended integral in the abstract theory of integration is relatively easy to prove. The situation, however, is quite different in the more elementary theories of integration. For instance, in the theory of the Riemann integral the concept of a Riemann-integrable function and the value of its integral is usually defined at the outset before the basic properties of the interval function, which measures the length of an interval, such as countable additivity or even additivity have been established. This is the main reason why Arzelà's theorem for the Riemann integral has the reputation of being difficult to prove without using results from the theory of Lebesgue measure. On the other hand, in the theory of Lebesgue integration the countable additivity property of the Lebesgue measure is one of the first basic results which are established. The Arzelà-Lebesgue dominated convergence theorem follows then rather easily. This state of affairs may account for the fact that the search for an "elementary proof", roughly meaning, independent of the theory of Lebesgue measure, for Arzelà's theorem is still on. A number of elementary proofs were published by F. Riesz [15] in 1917, by L. Bieberbach [4] and E. Landau [10] in 1918, by F. Hausdorff [9] and H. S. Carslaw [6] in 1927, by H. A. Lauwerier [11] in 1949, by J. D. Weston [16] in 1951, by W. F. Eberlein [8] in 1957, and by the present author [12] in 1961, respectively. Incidentally, in 1897, independently of C. Arzelà, W. F. Osgood [13] rediscovered Arzelà's theorem for continuous functions.

A few words concerning the known elementary proofs seem to be in order. L. Bieberbach gave a new and more elementary proof of Theorem B, and showed once more how to derive Arzelà's theorem from Theorem B. It seems that Arzelà's original proof of Theorem B (see [1], pp. 532–537) contained a gap which he filled later (see [1], pp. 596–599). A more detailed account of Arzelà's investigations can be found in [2]. E. Landau [10] gave an elementary and short proof that Theorem B implies Theorem A, thereby improving in part, Bieberbach's proof for Arzelà's theorem. F. Riesz [15] was the first to give a real elementary proof of Arzelà's theorem for continuous functions. He based his proof on Dini's uniform convergence theorem for monotone sequences of continuous functions rather than on Theorem B. F. Hausdorff [9] showed that Dini's theorem could also be used to obtain Arzelà's theorem for Riemann-integrable functions. But Hausdorff's proof seems to contain an error, which we shall discuss in more detail below. In [6], H. S. Carslaw presents his own version of the Bieberbach-Landau proof, which he remarks had gone unnoticed until that time in the English speaking world. In a footnote in the same article ([6], p. 438) Carslaw asks whether there exists also an elementary proof of a generalization of Arzelà's theorem due to W. H. Young [17]. An affirmative answer to this question is presented in the final section of the present article. H. A. Lauwerier [11] uses a form of Egoroff's theorem, but where he refers to Jordan content he really means Lebesgue measure. From a pair of inequalities for upper and lower integrals combined with an argument which could be used to prove Theorem B, J. D. Weston [16] obtains still another proof of Arzelà's theorem. W. F. Eberlein [8] proves Arzelà's theorem for Radon measures, defined on the space of real continuous functions on a compact Hausdorff space. Eberlein's proof is completely different from the proofs we have discussed so far. It is geometric in nature in that it is based on the parallelogram law and the minimal distance property for convex sets in inner product spaces. It is strongly recommended for study to the interested reader. In [12], the present author proves Arzelà's theorem for the abstract Riemann integral. It rests on a simplified version of a modification, due to I. Amemiya, of a technique used by F. Riesz in [15].

Despite the availability of this variety of elementary proofs for Arzelà's theorem, the present author finds that in most textbooks on analysis, whose authors have chosen to treat the Riemann integral rather than the Lebesgue integral, Arzelà's theorem is not mentioned, or, if it is mentioned, it is rarely accompanied by a correct proof or by any proof at all. In view of this, we should like to make one more attempt to show that Arzelà's theorem for the Riemann integral can be proved in an elementary fashion. By searching through the literature the author discovered that his new proof is essentially the same as Hausdorff's proof published in 1927. But with the important exception that at one point, where Hausdorff gives an incorrect inductive argument, the present author gives a simple direct argument which constitutes the main part of the proof of Lemma 2.2 below.

2. An elementary proof of Arzelà's theorem. Any proof of Arzelà's theorem depends in no small measure on how the Riemann integral is introduced. We shall assume in the rest of the paper, that a bounded real function on a bounded and closed interval is Riemann integrable if its lower Darboux integral is equal to its upper Darboux integral, and that the value of its integral is equal to the common value of its lower and upper integrals. Furthermore, we assume that the reader is familiar with the DINI UNIFORM CONVERGENCE THEOREM: *Each monotone sequence of continuous functions that converges pointwise to a continuous function on a bounded and closed interval is uniformly convergent.* We shall also use the following notation: $[a, b]$ denotes the bounded and closed interval $\{x: a \leqq x \leqq b\}$; $B[a, b]$ denotes the family of all bounded functions on $[a, b]$; $C[a, b]$ denotes the family of all continuous functions on $[a, b]$; $R[a, b]$ denotes the family of all Riemann-integrable functions on $[a, b]$;

$$\underline{\int_a^b} f(x)dx \quad \text{and} \quad \overline{\int_a^b} f(x)dx$$

denote the lower and upper Darboux integrals of a function $f \in B[a, b]$ respectively. By a step function we mean a finite linear combination of characteristic functions of intervals of finite length.

The proof of Arzelà's theorem will be based on the following two lemmas:

(2.1) LEMMA. *For each $0 \leqq f \in B[a, b]$ and for each $\epsilon > 0$, there exists a continuous function $g \in C[a, b]$ satisfying $0 \leqq g \leqq f$ and*

$$\underline{\int_a^b} f(x)dx \leqq \int_a^b g(x)dx + \epsilon.$$

Proof. From the definition of the lower integral it follows that for each $\epsilon > 0$ there exists a step function s on $[a, b]$ satisfying $0 \leqq s \leqq f$ and

$$\underline{\int_a^b} f(x)dx \leqq \int_a^b s(x)dx + \epsilon/2.$$

It is easy to see that s can be transformed into a trapezoidal function g, such that $0 \leqq g \leqq s$ and $\int_a^b s(x)dx \leqq \int_a^b g(x)dx + \epsilon/2$. Hence, there exists a continuous function $g \in C[a, b]$ satisfying $0 \leqq g \leqq f$ and

$$\underline{\int_a^b} f(x)dx \leqq \int_a^b g(x)dx + \epsilon;$$

and the proof is finished.

(2.2) LEMMA. *Let $\{f_n\}$ be a decreasing sequence of bounded functions on*

$[a, b]$. *If* $\lim_{n\to\infty} f_n(x) = 0$ *for all* $x \in [a, b]$, *then*

$$\lim_{n\to\infty} \underline{\int_a^b} f_n(x)dx = 0.$$

Proof. It follows from (2.1) that for each $\epsilon > 0$ and for each n, there exists a continuous function $g_n \in C[a, b]$ such that $0 \leqq g_n \leqq f_n$ and

$$\underline{\int_a^b} f_n(x)dx \leqq \int_a^b g_n(x)dx + \epsilon/2^n.$$

For each n, we set $h_n = \min(g_1, g_2, \cdots, g_n)$. Then $0 \leqq h_n \leqq g_n \leqq f_n$, $h_n \in C[a, b]$, and the sequence $\{h_n\}$ decreases to zero everywhere on $[a, b]$. Hence, by Dini's uniform convergence theorem, the sequence $\{h_n\}$ converges uniformly to zero on $[a, b]$, and consequently $\lim \int_a^b h_n(x)dx = 0$. The proof of the lemma will be finished if the following inequalities are established. For each n,

$$0 \leqq \underline{\int_a^b} f_n(x)dx \leqq \int_a^b h_n(x)dx + \epsilon(1 - (1/2^n)). \tag{2.3}$$

To this end, we shall first prove the following inequalities. For each n,

$$0 \leqq g_n \leqq h_n + \sum_{i=1}^{n-1} (\max(g_i, \cdots, g_n) - g_i). \tag{2.4}$$

The inequalities (2.4) follow easily by observing that for each $1 \leqq i \leqq n$,

$$\begin{aligned} 0 \leqq g_n = g_i + (g_n - g_i) &\leqq g_i + (\max(g_i, \cdots, g_n) - g_i) \\ &\leqq g_i + \sum_{i=1}^{n-1} (\max(g_i, \cdots, g_n) - g_i), \end{aligned}$$

so (2.4) follows. From $\max(g_i, \cdots, g_n) \leqq \max(f_i, \cdots, f_n) = f_i$ it follows that

$$\underline{\int_a^b} f_i(x)dx \geqq \int_a^b (\max(g_i, \cdots, g_n) - g_i)dx + \int_a^b g_i(x)dx,$$

so

$$\int_a^b (\max(g_i, \cdots, g_n) - g_i)dx \leqq \underline{\int_a^b} f_i(x)dx - \int_a^b g_i(x)dx \leqq \epsilon/2^i$$

for $i = 1, 2, \cdots, n$. Hence, by (2.4), for each n,

$$\int_a^b g_n(x)dx \leqq \int_a^b h_n(x)dx + \sum_{i=1}^{n-1} \epsilon/2^i = \int_a^b h_n(x)dx + \epsilon(1 - (1/2^{n-1})). \tag{2.5}$$

Finally, $\underline{\int_a^b} f_n(x) \leqq \int_a^b g_n(x)dx + \epsilon/2^n$ and (2.5) imply (2.3), and the proof is finished. ‾

REMARK. The reader does well to observe that Lemma 2.2 for Riemann integrable functions is already Arzelà's theorem for monotone sequences.

We shall now turn to the proof of Arzelà's theorem.

To this end, it is no loss in generality to assume that $0 \leqq f_n(x) \leqq M$ for all n and for all $x \in [a, b]$ and that $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for all $x \in [a, b]$. For each n, and for each $x \in [a, b]$, we set $p_n(x) = \sup_{k\geqq 0} (f_{n+k}(x))$. Then $0 \leqq f_n \leqq p_n$ and the sequence $\{p_n\}$ decreases everywhere to zero on $[a, b]$. Indeed, $0 = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \lim_{x\to\infty} p_n(x)$ for all $x \in [a, b]$. Hence, by Lemma 2.2,

$$\lim_{n\to\infty} \underline{\int_a^b} p_n(x)dx = 0,$$

and so, $0 \leqq \lim_{n\to\infty} \int_a^b f_n(x)dx \leqq \lim_{n\to\infty} \underline{\int_a^b} p_n(x)dx = 0$, that is, $\lim_{n\to\infty} \int_a^b f_n(x)dx = 0$, and the proof is finished.

One may perhaps feel that the above proof follows too closely the corresponding proof of the dominated convergence theorem for the Lebesgue integral and that an elementary proof of Arzelà's theorem should deal with Riemann integrable functions only. We will show that this is possible. It is clear that Lemma 2.1 can be shown to hold for Riemann integrable functions in exactly the same way. With respect to Lemma 2.2 the situation is somewhat different. For Riemann integrable functions we can avoid the inequalities (2.4) and prove the inequalities (2.3) directly as follows: We shall use the same notation as in the proof of Lemma 2.2 with the extra hypothesis that the functions f_n are Riemann integrable. Since the sequence $\{f_n\}$ is decreasing, we have for each n, $f_n = \min(f_1, \cdots, f_n)$, and so

$$0 \leqq f_n - h_n = \min(f_1, \cdots, f_n) - \min(g_1, \cdots, g_n) \leqq \sum_{i=1}^{n} (f_i - g_i).$$

Hence, for each n,

$$\begin{aligned} 0 &\leqq \int_a^b f_n(x)dx - \int_a^b h_n(x)dx \leqq \sum_{i=1}^{n} \int_a^b (f_i(x) - g_i(x))dx \leqq \sum_{i=1}^{n} \epsilon/2^i \\ &= \epsilon(1 - (1/2^n)). \end{aligned}$$

Since the lower integral is not subadditive but superadditive, the above method cannot be used to prove Lemma 2.2. That is why we had to introduce the inequalities (2.4) to obtain (2.3). It is also this point, where Hausdorff's proof is in error.

Having established Lemma 2.2 for Riemann integrable functions, we have, in fact, proven Arzelà's theorem for monotone sequences. The next question which we have to answer is whether we can deduce Arzelà's theorem directly from its special case for monotone sequences. It is not without interest that this is indeed true. This fact is contained in the paper by F. Riesz [15] as well as in the paper by W. F. Eberlein [8]. For the sake of completeness we shall show how this can be done. In order to bring out more dramatically that Arzelà's theorem is a logical consequence from the special case for monotone sequences, we shall adopt the following abstract setting:

Let X be a non-empty set, and let L be a linear space of real functions defined on X, satisfying $f \in L$ implies $|f| \in L$. The latter condition implies that for every finite set of elements $\{f_1, \cdots, f_n\}$ of L, $\max(f_1, \cdots, f_n) \in L$ and min $(f_1, \cdots, f_n) \in L$. A positive linear functional I on L is called an **integral** whenever I has the following property:

(2.6) *If* $0 \leqq f_n \in L$ *for each* n, *and the sequence* $\{f_n\}$ *decreases to zero everywhere on* X, *then* $\lim_{n\to\infty} I(f_n) = 0$.

It is obvious that (2.6) is Lemma 2.2 for I and L. We shall now show that (2.5) implies the following (abstract) Arzelà-type theorem:

(2.7) THEOREM. *Let* $f \in L$ *be the limit of an everywhere on* X *convergent sequence* $\{f_n\}$ *of elements of* L. *If there exists an element* $0 \leqq g \in L$ *satisfying* $|f_n(x)| \leqq g(x)$ *for all* $x \in X$ *and for all* n, *then for every integral* I *on* L *we have* $\lim_{n\to\infty} I(|f_n - f|) = 0$. *In particular*, $\lim_{n\to\infty} I(f_n) = I(f)$.

Proof. By considering the sequence $\{|f_n(x) - f(x)|\}$, which satisfies $|f_n(x) - f(x)| \leqq |f(x)| + g(x)$ for all $x \in X$ and for all n, where $|f| + g \in L$, we may assume without loss of generality that $f_n(x) \geqq 0$ for all $x \in X$ and for all n and that $f(x) = 0$ for all $x \in X$. Following F. Riesz [15], we set $g_{n,m} = \max(f_n, f_{n+1}, \cdots, f_m)$ for each pair of indices $m \geqq n$. Then $0 \leqq g_{m,n} \in L$ and $0 \leqq g_{m,n} \leqq g$ for all $m \geqq n$. Furthermore, for each n, the sequence $\{g_{m,n}\}$, and consequently, the sequence $\{I(g_{m,n})\}$ is increasing and bounded in $m > n$. Hence, for each $\epsilon > 0$ and for each n there exists an index $m_n > n$ such that $m_n < m_{n+1}$ and

$$(2.8) \qquad 0 \leqq I(g_{n,k}) - I(g_{n,m}) \leqq \epsilon/2^n,$$

for all $k \geqq m_n$. For the sake of simplicity we set $u_n = g_{n,m_n}$. Then

$$0 \leqq \limsup_{n\to\infty} u_n(x) \leqq \limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x) = 0$$

for all $x \in X$ implies that $\lim_{n\to\infty} u_n(x) = 0$ for all $x \in X$. If we apply the inequalities (2.4) to the sequence $\{u_n\}$, we obtain for each n,

$$(2.9) \qquad 0 \leqq f_n \leqq u_n \leqq \min(u_1, \cdots, u_n) + \sum_{i=1}^{n-1} (\max(u_i, \cdots, u_n) - u_i).$$

Since $\max(u_i, \cdots, u_n) - u_i = \max(f_i, \cdots, f_{m_n}) - u_i = g_{i,m_n} - g_{i,m_i}$, and $m_n > m_i$ for $n > i$, we conclude that $I((\max(u_i, \cdots, u_n) - u_i)) < \epsilon/2^i$ for $1 \leqq i \leqq n$, and so, by (2.9), for each n,

$$(2.10) \qquad 0 \leqq I(f_n) \leqq I(\min(u_1, \cdots, u_n)) + \epsilon(1 - (1/2)^{n-1}).$$

From $\lim_{n\to\infty} u_n(x) = 0$ for all $x \in X$, it follows that the sequence $\{\min(u_1, \cdots, u_n)\}$ decreases everywhere to zero on X. Hence, by hypothesis, $\lim_{n\to\infty} I(\min(u_1, \cdots, u_n)) = 0$, and finally, using (2.10), we obtain that $\lim_{n\to\infty} I(f_n) = 0$, and the proof is finished.

REMARK. It is not difficult to convince oneself that the above proofs can be so modified as to obtain Arzelà's theorem for Riemann integrable functions of several variables. For the Riemann-Stieltjes integral, Arzelà's theorem also holds provided it is introduced in such a way that (2.1) and (2.2) hold. For this purpose, it is necessary and sufficient that the Stieltjes measure for intervals is defined in such a way that it is a countably additive interval function. In that case, the proofs of (2.1) and (2.2) remain the same. Conversely, (2.2) implies the countable additivity property of the Stieltjes measure.

3. Fatou's lemma for the Riemann integral. In the theory of Lebesgue integration, Fatou's lemma plays an important role. The analogous result for the Riemann integral will follow easily from the following lemma:

(3.1) LEMMA. *Let* $0 \leqq f \in R[a, b]$ *be the limit of an everywhere convergent sequence* $\{f_n\}$ *of non-negative Riemann integrable functions on* $[a, b]$. *Then*

$$\lim_{n\to\infty} \int_a^b (f(x) - f_n(x))^+ dx = 0,$$

where $(f(x) - f_n(x))^+ = \max(f(x) - f_n(x), 0)$ *for all* $x \in [a, b]$.

Proof. Since the functions f_n, f are non-negative it follows that $f(x) - f_n(x) \leqq f(x)$ for all $x \in [a, b]$. Hence, $(f(x) - f_n(x))^+ \leqq f(x)$; and the result follows from Arzelà's theorem.

(3.2) THEOREM. (Fatou's lemma for the Riemann integral). *Under the same hypotheses of* (3.1), *we have*

$$0 \leqq \int_a^b f(x)dx \leqq \liminf_{n\to\infty} \int_a^b f_n(x)dx.$$

Proof. Observe that $f = (f - f_n) + f_n \leqq (f - f_n)^+ + f_n$ for all n. Hence, by (3.1),

$$\int_a^b f(x)dx \leqq \liminf_{n\to\infty} \left(\int_a^b (f(x) - f_n(x))^+ dx + \int_a^b f_n(x)dx \right) = \liminf_{n\to\infty} \int_a^b f_n(x)dx,$$

and the proof is finished.

From (3.1) we can also deduce the following result supplementing Fatou's lemma:

(3.3) THEOREM. *Under the same hypotheses of* (3.1), *and* $\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$, *we have* $\lim_{n\to\infty} \int_a^b |f(x) - f_n(x)| dx = 0$.

Proof. From (3.1), the new hypothesis, and

$$f(x) - f_n(x) = (f(x) - f_n(x))^+ - (f(x) - f_n(x))^-$$

for all n, it follows that $\lim_{n\to\infty} \int_a^b (f(x) - f_n(x))^- dx = 0$, where $(f(x) - f_n(x))^- = \max(-(f(x) - f_n(x)), 0)$, $x \in [a, b]$. Hence

$$\lim_{n\to\infty} \int_a^b | f(x) - f_n(x) | dx$$
$$= \lim_{n\to\infty} \left(\int_a^b (f(x) - f_n(x))^+ dx + \int_a^b (f(x) - f_n(x))^- dx \right) = 0,$$

and the proof is finished.

4. W. H. Young's extension of Arzelà's theorem. In [**17**], Test 6, p. 316, W. H. Young gave an extension of Arzelà's theorem for the Lebesgue integral, which appears as a problem about term-by-term integration in [**5**], Example 22, p. 144. In [**6**], Carslaw asks whether there exists a simple proof for Young's result. Since the result of Young is interesting in itself, we shall present it here supplied with an elementary proof.

(4.1) THEOREM (W. H. Young). *Assume f_n, g_n, and $h_n \in R[a, b]$ for each n and that the sequences $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ converge everywhere on $[a, b]$ to the Riemann integrable functions f, g, and h, respectively. If $h_n \leqq f_n \leqq g_n$ for each n, and passage of the limit under the integral sign for the sequences $\{g_n\}$, $\{h_n\}$ is possible, that is, $\lim_{n\to\infty} \int_a^b g_n(x)dx = \int_a^b \lim_{n\to\infty} g_n(x)dx = \int_a^b g(x)dx$ and $\lim_{n\to\infty} \int_a^b h_n(x)dx = \int_a^b \lim_{n\to\infty} h_n(x)dx = \int_a^b h(x)dx$, then the same holds for the sequence $\{f_n\}$, that is, $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$.*

For the case that $h_n(x) = -M$ and $g_n(x) = M$ for all n and for all $x \in [a, b]$, where M is a positive constant, Young's theorem reduces to Arzelà's theorem.

We shall now show that Young's theorem follows from Arzelà's theorem, Theorem 3.3 and the following lemma:

(4.2) LEMMA. *If $0 \leqq u$, v, and $w \in R[a, b]$ satisfy $0 \leqq u \leqq v + w$, then u can be written in the form $u = u_1 + u_2$, where $0 \leqq u_1 \leqq v$, $0 \leqq u_2 \leqq w$, and $u_1, u_2 \in R[a, b]$.*

Proof. Let $u_1 = \min(u, v)$ and let $u_2 = u - u_1$. Then $0 \leqq u_1 \leqq v$ and $0 \leqq u_2 = u - u_1 = \min(u, v+w) - \min(u, v) \leqq v + w - v = w$. Since $\min(u, v) = \frac{1}{2}(u + v - |u - v|)$ it follows that $u_1 \in R[a, b]$, and so also $u_2 = u - u_1 \in R[a, b]$, finishing the proof.

We shall now turn to the proof of Young's theorem. From $h_n \leqq f_n \leqq g_n$ it follows that $0 \leqq f_n - h_n \leqq g_n - h_n = g_n - h_n - (g - h) + (g - h)$. Observe that the sequence $\{g_n - h_n\}$ satisfies the hypotheses of Theorem 3.3, and so, by setting $u_n = |g_n - h_n - g + h|$ for each n, we conclude that

$$\lim_{n\to\infty} \int_a^b u_n(x)dx = 0. \tag{4.3}$$

Then $0 \leqq f_n - f - h_n + f \leqq u_n + |g - h|$ for all n. Hence, from (4.1) it follows that $0 \leqq f_n - f - h_n + f = v_n + w_n$ for all n, where $0 \leqq v_n$, $w_n \in R[a, b]$, $0 \leqq v_n \leqq u_n$, and $0 \leqq w_n \leqq |g - h|$ for all n. Since $\lim_{n\to\infty} u_n(x) = 0$ for all $x \in [a, b]$ it follows that $\lim_{n\to\infty} w_n(x) = f(x) - h(x)$ for all $x \in [a, b]$. Then $0 \leqq w_n \leqq |g - h| \in R[a, b]$ for all n implies, by Arzelà's theorem,

$$\lim_{n\to\infty} \int_a^b w_n(x)dx = \int_a^b f(x)dx - \int_a^b h(x)dx. \tag{4.4}$$

Hence, by (4.3) and (4.4) we have

$$
\lim_{n\to\infty}\left(\int_a^b (f_n(x)-f(x))dx-\int_a^b h_n(x)dx+\int_a^b f(x)dx\right) = \int_a^b f(x)dx-\int_a^b h(x)dx. \tag{4.5}
$$

From the hypothesis $\lim_{n\to\infty}\int_a^b h_n(x)dx=\int_a^b h(x)dx$ and (4.5) it follows finally that $\lim_{n\to\infty}\int_a^b (f_n(x)-f(x))dx$ exists and is equal to zero, and the proof is finished.

REMARK. The reader will have no difficulty in showing that if, in addition, the sequences $\{g_n\}$, and $\{h_n\}$ satisfy $\lim_{n\to\infty}\int_a^b |h_n(x)-h(x)|\,dx=0$ and

$$\lim_{n\to\infty}\int_a^b|g_n(x)-g(x)|\,dx=0,$$

then the sequence $\{f_n\}$ also has the property $\lim_{n\to\infty}\int_a^b |f_n(x)-f(x)|\,dx=0$.

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