Closed and Exact

The words *closed* and *exact* have many meanings. I will make up some terms that are private for this class.

Definition 1. *A vector field* $\mathbf{F}$ *is curl-closed if* $curl(\mathbf{F}) = 0$*.* $\mathbf{F}$ *is gradient-exact if* $\mathbf{F} = grad(f)$*.* $\mathbf{F}$ *is div-closed if* $div(\mathbf{F}) = 0$*.* $\mathbf{F}$ *is curl-exact if* $\mathbf{F} = curl(\mathbf{G})$*.*

Theorem 1. *On a box with faces parallel to the axis planes,*

$$\textit{curl-closed} \Longleftrightarrow \textit{gradient-exact}, \tag{1}$$

$$\textit{div-closed} \Longleftrightarrow \textit{curl-exact}. \tag{2}$$

Proof. I will only prove statement (2).

First, if $\mathbf{F} = curl(\mathbf{G})$ where $\mathbf{G} = (U, V, W)$. Then

$$div(\mathbf{F}) = (W_y - V_z)_x - (W_x - U_z)_y + (V_x - U_y)_z \tag{3}$$

$$= 0. \tag{4}$$

Next let $\mathbf{F} = (P, Q, R)$ and suppose $P_x + Q_y + R_z = 0$. Suppose there is vector field $\mathbf{G} = (U, V, W)$ so that $\mathbf{F} = curl(\mathbf{G})$. If we add $grad(f)$ to $\mathbf{G}$ then since $curl(grad(f)) = 0$, it is still true that $\mathbf{F} = curl(\mathbf{G})$. Now we can always choose f so that $f_z = -W$, in which case the z-component of $\mathbf{G} + grad(f)$ is 0. In other words we can assume that $W = 0$. Now our requirements become

$$P = -V_z, \tag{5}$$

$$Q = U_z, \tag{6}$$

$$R = V_x - U_y. \tag{7}$$

We solve the first two equations by taking any z-antiderivative of $-P$ for V and any z-antiderivative of Q for U. In symbolic form

$$V(x, y, z) = -\int_{z_0}^{z} P(x, y, t)dt + \phi(x, y), \tag{8}$$

$$U(x, y, z) = \int_{z_0}^{z} Q(x, y, t)dt + \psi(x, y). \tag{9}$$

Where we have let $\phi(x, y) = V(x, y, z_0)$ and $\psi(x, y) = U(x, y, z_0)$. We can do this on a box. Now we need to solve (7). But (7) is

$$R(x, y, z) = -\int_{z_0}^{z} (Q_y + P_x)dt + \phi_x - \psi_y \tag{10}$$

$$= R(x, y, z) - R(x, y, z_0) + \phi_x - \psi_y. \tag{11}$$

We can solve this equation by letting $\psi = 0$ and chosing any solution of $\phi_x(x, y) = R(x, y, z_0)$. □

Example :
Let $\mathbf{F} = (3x^2y, -xy^2, -4xyz)$. Then check that $div(\mathbf{F}) = 0$. Equations 5,6,7 become

$$3x^2y = -V_z, \tag{12}$$
$$-xy^2 = U_z, \tag{13}$$
$$-4xyz = V_x - U_y. \tag{14}$$

Following the proof of the theorem we find that

$$U = -zxy^2 + \phi(x, y), \tag{15}$$
$$V = -3x^2yz + \psi(x, y). \tag{16}$$

Let $\psi = 0$ and then find that equation 14 is

$$-\phi_y = 0,$$

so we also choose $\phi = 0$. The solution is then

$$U = -zxy^2, \tag{17}$$
$$V = -3x^2yz, \tag{18}$$
$$W = 0. \tag{19}$$