Power Series

February 9, 2011

This note is an exposition of basic facts about power series. First a reminder about $\overline{\lim}$.

Definition 1. *Let $\{a_n\}$ be a sequence of real numbers.*

$$\overline{\lim}\, a_n = \lim_{m\to\infty} \sup\{a_k : k \geq m\}.$$

Proposition 1. *$\ell = \overline{\lim}\, a_n$ if and only if*

1. *For every $\epsilon > 0$ there is an N so that $\sup\{a_j : j \geq N\} \leq \ell + \epsilon$.*
2. *For every $\epsilon > 0$ there $a_n \geq \ell - \epsilon$ for infinitely many n.*

Proof. Let $s = \overline{\lim}\, a_n$.

Suppose ℓ is a number satisfying (1) and (2). Then (2) implies that $\sup\{a_k : k \geq m\} \geq \ell - \epsilon$ since there is always an element of $\{a_k : k \geq m\}$ that is at least as big as $\ell - \epsilon$. So $s \geq \ell - \epsilon$. Also by (1) $s \leq \sup\{a_j : j \geq N\} \leq \ell + \epsilon$. So $\ell - \epsilon \leq s \leq \ell - \epsilon$ and since ϵ is arbitrary, $s = \ell$.

On the other hand $\sup\{a_j : j \geq N\} \leq s + \epsilon$ since $\sup\{a_j : j \geq N\}$ decreases to s. And there must be infinitely many n so that $a_n \geq s - \epsilon$ or else $s \leq s - \epsilon$ which clearly cannot happen. $\square$

Theorem 1. *Let $\ell = \overline{\lim}\, |a_n|^{\frac{1}{n}}$. Let $R = \frac{1}{\ell}$.*

1. *If $|x| < R$, $\sum_0^\infty a_n x^n$ converges absolutely.*
2. *If $|x| > R$, $\sum_0^\infty a_n x^n$ diverges.*

Proof.

1. If $|x| < R = \frac{1}{\ell}$ then choose ϵ so small that $\rho = |x|\ell + \epsilon < 1$. Then $|x||a_n|^{\frac{1}{n}} \leq \rho$ if n is large. Hence $|a_n x^n| \leq \rho^n$ and by comparison $\sum_0^\infty a_n x^n$ converges absolutely.
2. If $|x| > R$, then $|x|\ell > 1$ and now choose $\epsilon > 0$ so that $r = |x|\ell - \epsilon > 1$ Then for infinitely many n, $|x||a_n|^{\frac{1}{n}} \geq r > 1$. Hence $\sum_0^\infty a_n x^n$ diverges.

$\square$

Definition 2. *$R = \frac{1}{\ell}$, where $\ell = \overline{\lim}\, |a_n|^{\frac{1}{n}}$ is called the* ***radius of convergence*** *of $\sum_0^\infty a_n x^n$.*

Theorem 2. *$R = \sup\{|x| : \{a_n x^n : n = 1, 2, \dots\}$ is bounded$\}$.*

Proof. Let $S = \sup\{|x| : \{a_n x^n : n = 1, 2, \dots\}$ is bounded$\}$. If $|x| > S$ then $\{a_n x^n : n = 1, 2, \dots\}$ is not bounded. Hence $\sum_0^\infty a_n x^n$ does not converge because the terms do not go to 0. If $|x| < S$ then $|x| < S_0 < S$ and let $M = \sup\{|a_n S_0^n\}$. So $|a_n x^n| = |a_n S_0^n| \left(\frac{|x|}{S_0}\right)^n \leq M \left(\frac{|x|}{S_0}\right)^n$ and by comparison $\sum_0^\infty a_n x^n$ converges since $\frac{|x|}{S_0} < 1$. We have proved that if $|x| < S$, $\sum_0^\infty a_n x^n$ converges and if $|x| > S$, $\sum_0^\infty a_n x^n$ diverges. The same property is true of R. There can be only one number with this property. So $R = S$. $\square$

We still need to deal with the question of uniform convergence.

Theorem 3. *Let R be the radius of convergence of $\sum_0^\infty a_n x^n$. Let $c < R$. Then $\sum_0^\infty a_n x^n$ converges uniformly on $[-c, c]$.*

Proof. Let $c < d < R$ and let $|x| \leq c$. Then $|a_n d^n| \leq M$ and $|a_n x^n| = |a_n d^n| \left(\frac{|x|}{d}\right)^n \leq |a_n d^n| \left(\frac{c}{d}\right)^n \leq M \left(\frac{c}{d}\right)^n$. (Notice it is not necessarily true that $|a_n R^n| \leq M$ so we need to squeeze d between c and R.) $\square$

So *every* power series converges uniformly on compact subsets of its interval of convergence.

Now $\overline{\lim}(n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}) = \overline{\lim}\, |a_n|^{\frac{1}{n}}$ so the radius of convergence of the series obtained by differentiating term-by-term is the same as the radius of convergence of the original series and hence the differentiated series also converges uniformly on compact subsets of the same interval of convergence as the original series.

Theorem 4. *If $f(x) = \sum_0^\infty a_n x^n$ converges on $I = (-R, R)$ then $f'(x)$ exists on I and $f'(x) = \sum_1^\infty n a_n x^{n-1}$.*

Corollary 1. *If $f(x) = \sum_0^\infty a_n x^n$, $a_n = \frac{f^{(n)}(0)}{n!}$.*