Abel's Test

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This note is an exposition of Abel's test on convergence of series.

Theorem 1. *Suppose $\sum_1^\infty b_n$ converges and that $\{a_n\}$ is a monotone bounded sequence. Then $\sum_1^\infty a_n b_n$ converges.*

Proof. Let $b_0 = 0$, $B_N = \sum_{k=0}^N b_k$. Then $b_n = B_n - B_{n-1}, n \geq 1$, hence

$$\begin{aligned}\sum_{k=1}^N a_k b_k &= \sum_{k=1}^N a_k(B_k - B_{k-1}) \\ &= B_1(a_1 - a_2) + B_2(a_2 - a_3) + \dots B_{N-1}(a_{N-1} - a_N) + a_N B_N \\ &= \sum B_k(a_k - a_{k+1}) + a_N B_N\end{aligned}$$

Since $\{a_n\}$ is monotone and bounded it converges; and $\{B_N\}$ converges since $\sum b_n$ converges. Hence $a_N B_N$ converges. We estimate $\sum B_k(a_k - a_{k+1})$. Since $\sum b_n$ converges, $|\sum b_n| \leq M$ for some M. Using the fact that $\{a_n\}$ is montone we get $\sum_1^N |B_k(a_k - a_{k+1}| \leq M \sum_1^N |a_k - a_{k+1}| = M|a_1 - a_{N+1}| \to M|a_1 - a|$, where $a_k \to a$. Hence $\sum B_k(a_k - a_{k+1}$ converges absolutely. □

Example 1. *Suppose $\sum a_n$ converges. Then $\sum n^{1/n} a_n$ converges and $\sum (1 + 1/n)^n a_n$ converges.*

Proof. It's easy to prove that $f(x) = x^{1/x}$ is decreasing for $x > e$, by computing the derivative of $(\log(x))/x$. Here is the proof that $(1 + 1/n)^n$ increases with n.

$$\begin{aligned}(1 + 1/n)^n &= 1 + 1 + \cdots + \binom{n}{p}\frac{1}{n^p} + \cdots + \frac{1}{n^n} \\ &= 1 + \cdots + (1 - 1/n)\dots(1 - (p-1)/n)\frac{1}{p!} + \cdots + \frac{1}{n^n} \\ (1 + 1/(n+1))^{n+1} &= 1 + \cdots + (1 - 1/(n+1))\dots(1 - (p-1)/(n+1))\frac{1}{p!} + \cdots + \frac{1}{(n+1)^{n+1}}.\end{aligned}$$

The last sum has one more (positive) term than the preceding term and the pth term of the last sum is larger than the preceding pth term since each factor is larger (subtract $k/(n+1)$ from 1 as opposed to subtracting k/n). Hence

$$(1 + \frac{1}{n+1})^{n+1} > (1 + \frac{1}{n}).$$

It's easy check using L'Hopital's rule that $(1 + 1/x)^x \to e$ as $x \to \infty$, so the sequence $(1 + 1/n)^n$ is monotone and bounded.

□