Riemann-Lebesgue Lemma

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The Riemann-Lebesgue lemma is quite general, but since we only know Riemann integration, I'll state it in that form.

Theorem 1. *Let f be Riemann integrable on $[a, b]$. Then*

$$\lim_{\lambda \to \pm\infty} \int_a^b f(t) \cos(\lambda t) dt = 0 \tag{1}$$

$$\lim_{\lambda \to \pm\infty} \int_a^b f(t) \sin(\lambda t) dt = 0 \tag{2}$$

$$\lim_{\lambda \to \pm\infty} \int_a^b f(t) e^{i\lambda t} dt = 0 \tag{3}$$

Proof. I will prove only the first statement. Since f is integrable, given $\epsilon > 0$, there is a partition $\{a = x_0, x_1, \dots, x_n = b\}$, so that $\epsilon/2 > \displaystyle\int_a^b f - \sum_1^n m_i \Delta x_i \geq 0$, where m_i is the minimum of f on $[x_{i-1}, x_i]$. But the sum can be written as $\sum_1^n m_i \Delta x_i = \int_a^b g$, where $g = \sum m_i \chi_{[x_{i-1}, x_i]}$, and the inequality takes the form

$$\epsilon/2 > \int_a^b (f - g) \geq 0.$$

Now we use the fact that $f - g \geq 0$ to get

$$\left| \int_a^b f(t) \cos(\lambda t) dt \right| \leq \left| \int_a^b (f(t) - g(t)) \cos(\lambda t) dt \right| + \left| \int_a^b g(t) \cos(\lambda t) dt \right| \tag{4}$$

$$\leq \int_a^b (f - g) + \left| (1/\lambda) \sum m_i \left(\sin(\lambda x_i) - \sin(\lambda x_{i-1}) \right) \right|. \tag{5}$$

The function g has been fixed. Take λ large enough that

$$\left| (1/\lambda) \sum m_i \left(\sin(\lambda x_i) - \sin(\lambda x_{i-1}) \right) \right| < \epsilon/2,$$

and we are done.

$\square$

This proof works nearly verbatim for Lebesgue integration and non-compact intervals.