Computing the Laplace Transform

We will give a technique for computing Laplace transforms once we know other Laplace transforms. This material is taken from [1]

Theorem 1. *[1] Suppose ϕ is piecewise continuous on $[0, \infty)$ and to make it simple, assume $|\phi(t)| \leq M, |\phi(t)| \leq M$ on $[0, \infty)$. Then it is true that $f(s) = \int_0^\infty e^{-st}\phi(t)dt$ converges absolutely and uniformly for $R \geq s \geq s_0 > 0$. Suppose also that $\int_0^1 \frac{|\phi(t)|}{t}$ exists. Then for $s > 0$*

$$\int_0^\infty e^{-st}\frac{\phi(t)}{t}dt = \int_s^\infty f(x)dx.$$

Proof. The integral $f(s) = \int_0^\infty e^{-st}\phi(t)dt$ converges absolutely and uniformly so we can integrate from $x = s_0$ to $x = R$ and change the order of integration.

$$\begin{aligned}
\int_{s_0}^R f(x)dx &= \int_0^\infty \phi(t)[\int_{s_0}^R e^{-xt}dx]dt \\
&= \int_0^\infty \frac{\phi(t)}{t}[e^{-s_0t} - e^{-Rt}]dt \\
&= \int_0^\infty \frac{\phi(t)}{t}e^{-s_0t}dt - \int_0^\infty \frac{\phi(t)}{t}e^{-Rt}dt.
\end{aligned}$$

An easy estimate proves that $\int_0^\infty \frac{\phi(t)}{t}e^{-Rt}dt \to 0$ as $R \to \infty$. Notice we have proved that $\int_s^\infty f(x)dx$ converges. □

Corollary 1.

$$\int_0^\infty e^{-st}\frac{\sin t}{t}dt = \frac{\pi}{2} - \arctan(s.$$

Proof. It is easy to compute

$$\int_0^\infty e^{-st}\sin(t)dt = \frac{1}{1+s^2},$$

by noticing that $\sin t = Im(e^{it})$. Then apply the theorem. □

References

[1] Widder, David , *Advanced Calculus*, Theorem 7, p 450; Prencice Hall, (1961).