Baire Category Theory

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This document will be evolving (like a blog) over time. It will include an introduction to Baire category theory and applications. To be explicit I will assume we are working in $\mathbb{R}$, but many of the statements and proofs will be correct in a more general context. The concept of countability is crucial in the definitions. I will start off with the definitions. Their meaning will be developed in the applications. There are many good references. I will often take proofs from John Oxtoby's book, *Measure and Category*, [1].

Definition 1. *A set E is* ***dense*** *in an interval I if for every subinterval $J \subset I$, $J \cap E \neq \varnothing$. A set E is* ***nowhere dense*** *if it is not dense in any interval.*

Remark 1. *If E is nowhere dense, then for every interval I there is some subinterval J of I such that $J \cap E = \varnothing$. This is equivalent to the statement that the closure of E, $\overline{E}$ has no interior. E is nowhere dense if and only if $\overline{E}$ is nowhere dense. If E is nowhere dense every open set must contain points in $\overline{E}^c$, so the complement of the closure of a nowhere dense set is dense. If the complement of the closure of E is dense then the E is nowhere dense.*

Suppose E_1 is nowhere dense. Then $\overline{E_1} \neq \mathbb{R}$, so there is a point $p \notin \overline{E_1}$ and hence a nonempty compact ball around p, B_1, that is disjoint from E_1. We can suppose the radius of B_1 is less than 1. Now take a second nowhere dense set E_2. Then B_1 is not contained in $\overline{E_2}$ so we can select a point in the interior of B_1 and a closed ball B_2 of radius less than $1/2$ inside B_1 that is disjoint from E_2. This ball will also be disjoint from E_1. Let E_j be a sequence of nowhere dense sets. We can construct a nested sequence of balls $B_{j+1} \subset B_j$ which are disjoint from $\cup E_j$ and by the nested interval theorem they have a non-empty intersection $\cap B_j$. We have proved that $\cup E_j \neq \mathbb{R}$. This is the Baire category theorem which we will now state.

Definition 2. *A set is of* ***first category*** *if it is a* ***countable*** *union of nowhere dense sets. For example $\mathbb{Q}$ is a set of first category. A set is of* ***second category*** *if it is not a set of first category. A* ***generic*** *set is the complement of a set of first category.*

Theorem 1 (**Baire Category Theorem**). *$\mathbb{R}$ is not of first category. If C is a set of first category then there are points not in C. A set of first category can not be all of $\mathbb{R}$. A generic set in $\mathbb{R}$ is dense.*

Proof. The only thing left to prove is the density statement. An examination of the proof shows that any open set contains a point not in $\overline{E_1}$ and hence a nonempty open set disjoint from $\overline{E_1}$. Continuing the proof we conclude that any open set contains a point of $(\cup E_j)^c$. In other words any neighborhood of any point contains a point of $(\cup E_j)^c$. This implies that $(\cup E_j)^c$ is dense. This also proves that $(\cup \overline{E_j})^c$ is dense. □

Corollary 1. *The intersection of a dense sequence of open sets is dense.*

Proof. Let G_j be open and $\overline{G_j} = \mathbb{R}$. Let $F_j = \mathbb{R} - G_j$. Then F_j is nowhere dense. It follows that $\cap_j G_j = (\cup_j F_j)^c$ is a generic set and hence dense. □

Remark 2. *The same proof shows that any interval (open, closed, or neither) is not a set of first category and it also proves that $\mathbb{R}^n$ is not of first category.*

We will use this result to prove some striking theorems. We also will use the following definitions.

Definition 3. *A subset E of $\mathbb{R}^n$ is a G_δ if $E = \cap_{n=1}^{\infty} G_n$ where each G_n is and open set. Notice the intersection is countable. A set B is an F_σ it is a countable union $B = \cup F_n$ where each F_n is closed.*

Proposition 1. *Closed and open sets are both G_δ's and F_σ's.*

Proof. It is clear that an open set is a G_δ. Every open set is a countable union of closed balls with rational centers and rational radii. Since the complement of an F_σ is a G_δ the result for closed sets follows. □

Proposition 2. *The set $\mathbb{Q}$ of rational numbers is an F_σ but not a G_δ.*

Proof. If $\mathbb{Q}$ were a G_δ, then the complement of $\mathbb{Q}$ is the countable union of closed sets $\mathbb{Q}^c = \cup G_n^c$. Since $\mathbb{Q}^c$ has no interior, $(G_n^c)^o = \varnothing$, where E^o denotes the interior of a set. But we have written $\mathbb{R} = \mathbb{Q} \cup (\cup G_n^c)$ as a countable union of nowhere dense sets and this contradicts the Baire category theorem. □

Let f be a function defined on $\mathbb{R}^n$. We define the oscillation of f on a set W by $o_W(f) = \sup_W(f) - \inf_W(f)$, where $\sup_W(f)$, $\inf_W(f)$ are the sup and inf of f on W. Let $B_n(x) = \{y : \|y - x\| < 1/n$ and define $o_f(x) = \lim_{n\to 0} o_{B_n(x)}(f)$.

Proposition 3. *f is continuous at a if and only if $o_f(a) = 0$.*

Proof. This is true by the $\epsilon - \delta$ definition of continuity. □

Proposition 4. *For every number r, $\{x | o_f(x) < r\}$ is an open set (and $\{x | o_f(x) \geq r\}$ is closed).*

Proof. If $o_f(x) < r$ then for some n, $o_{B_n(x)}(f) < r$ and then for all $y \in B_n(x)$, $o_f(y) < r$. □

Theorem 2. *Let $D(f)$ be the set of discontinuity points of f. Then $D(f) = \{x | \cup_k \{o_f(x) \geq 1/k\}$ and hence $D(f)$ is an F_σ.*

Theorem 3. *Let $F = \cup_{n=1}^{\infty} F_n$ be an F_σ, where each F_n is closed. Then there is a function f with $D(f) = F$.*

Proof. (Taken from [1].) We will assume $F \neq \mathbb{R}^n$. Let A_n be the set of rational points in F_n^o (might be empty) and let
$f_n = \chi_{F_n \setminus A_n} = \chi_{F_n} - \chi_{A_n}$. Then

$$o_{f_n}(x) = \begin{cases} 1 \text{ if } x \in F_n \\ 0 \text{ if } x \notin F_n. \end{cases}$$

It is important that F_n be closed for this to be correct. This implies $D(f_n) = F_n$. Now let $a_n = 1/n!$ and $f = \sum_1^{\infty} a_n f_n$. We claim that $a_n > \sum_{m>n} a_m$. This is an easy thing to check. By the Weierstrass M-test, the series converges uniformly. Since each f_n is continuous on $\mathbb{R} - F$, f is continuous on $\mathbb{R} - F$. Now we estimate the oscillation of f at each point of F. On F_1

$$0 \leq \sum_2^{\infty} a_m f_m(x) \leq \sum_2^{\infty} a_m < a_1.$$

In each neighborhood of a point x of F_1^o the maximum of f is at least a_1 and the minimum is less than or equal to $\sum_2^{\infty} a_m$. So $o_f(x) \geq a_1 - \sum_{m\geq 2} a_m > 0$. If $x \in \partial F_1$ then $f_1(x) = 1$ and the maximum of f is at least a_1. Again, the minimum is no more than $\sum_2^{\infty} a_m$, so $o_f(x) > 0$. A similar argument works at each point of $F_n - F_{n-1}$ to prove that $o_f(x) > 0, x \in F$, thus $D(f) = F$. □

Now let's see what we can say about limits of continuous functions.

Theorem 4. *Suppose* f_n *is continuous on* I *for each* n *and suppose* $\lim_{n\to\infty} f_n(x) = f(x), x \in I$, *where* I *is an interval. Then* $D(f)$ *is a set of first category.*

Proof. Let $c > 0$. First, we prove that $E_c = \{x : o_f(x) \geq c\}$ is nowhere dense. If we know this, then $D(f) = \cup_k \{x | o_j(x) \geq 1/k\}$ is of first category.

We will show that any compact subinterval in I has a subinterval that is not a subset of E_c. Choose d with $3d < c$, let K be a compact subinterval of I, and let

$$F_n = \cap_{k,l \geq n} \{x \in K : |f_k(x) - f_l(x)| \leq d\}.$$

F_n is the intersection of closed sets and hence closed. Convergence on I and the Cauchy criterion implies that $\cup_n F_n = K$. By the Baire category theorem, not every F_n can be nowhere dense. Then for some n there is a compact interval J such that $J \subset F_n$. Now fix $l = n$ and let $k \to \infty$ to get $|f(x) - f_n(x)| \leq d$ for all $x \in J$. Since f_n is continuous there is a $\delta > 0$ so that $|f_n(x) - f_n(y)| < d$ for $|x - y| < \delta$ (uniform continuity of f_n on J). Then $|f(x) - f(y)| < 3d$ when $|x - y| < \delta$. So if $x \in J$, $o_f(x) < 3d$ and hence $J \not\subset \{x : o_f(x) \geq c\}$. □

Corollary 2. *The set of discontiuities of a derivative is a set of first category. The set of points of continuity of a derivative is a generic set and hence dense. There are "lots" of points of continuity of a derivative.*

Proof. If f has a derivative, f is continuous. Then

$$f'(x) = \lim_{n\to\infty} \frac{f(x + 1/n)}{1/n},$$

so f' is a limit of a sequence of continuous functions. We have to be more careful on an interval since if f is defined on an interval I, $f(x+1/n)$ might not be defined on that interval. If f has a derivative on a closed interval, we can extend f linearly beyond the endpoints to a function that is differentiable everywhere. If I is not closed, we can restrict f to a sequence of closed sub intervals that exhaust I and conclude that the set $D(f')$ is a countable union of sets of first category and hence is of first category. □

References

[1] John Oxtoby, *Measure and Category*, Springer-Verlag, 1996.